TECHNISCHE UNIVERSITÄT DRESDEN

Institute for Statics und Dynamics of Structures

Fuzzy Time Series

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1 Description of fuzzy time series

2 Modelling of fuzzy time series

3 Forecasting of fuzzy time series

4 Examples

5 Conclusions


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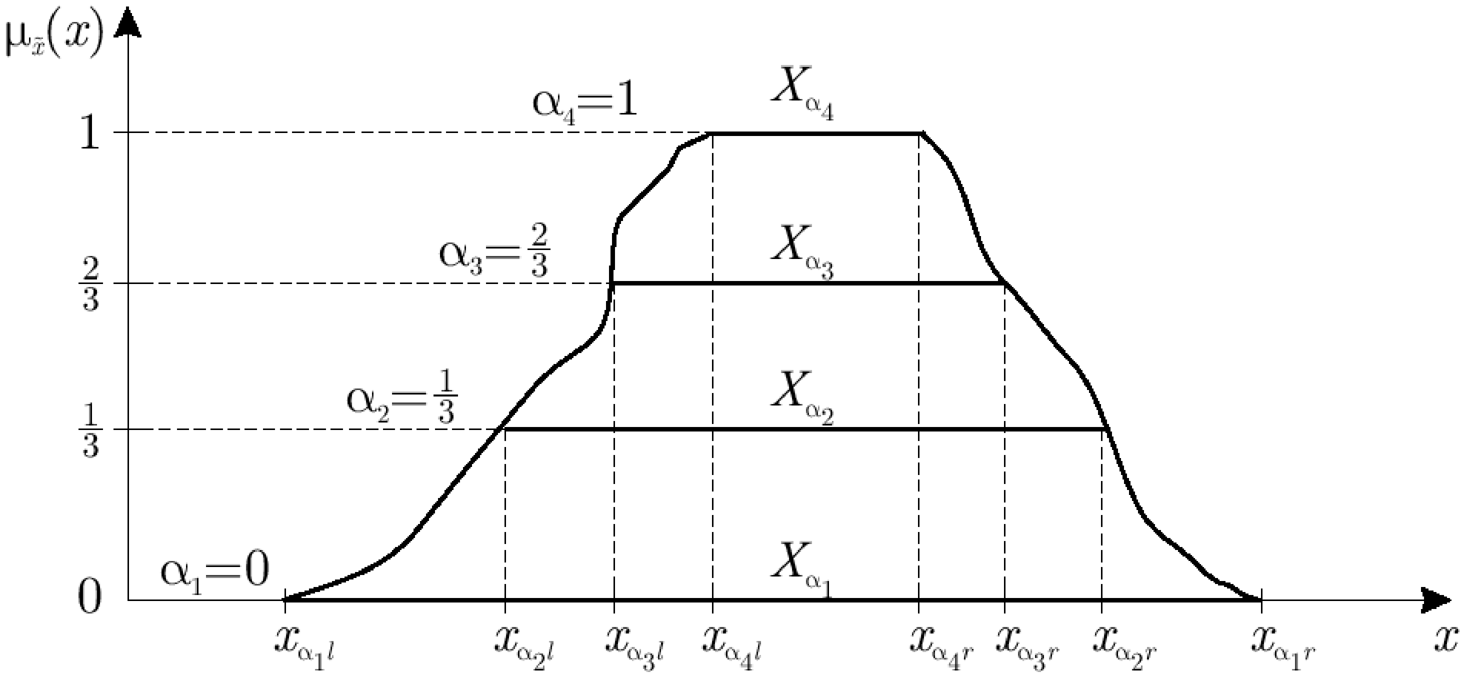
$\tilde{x}_\tau$
time series with fuzzy data
forecasting
?
τ

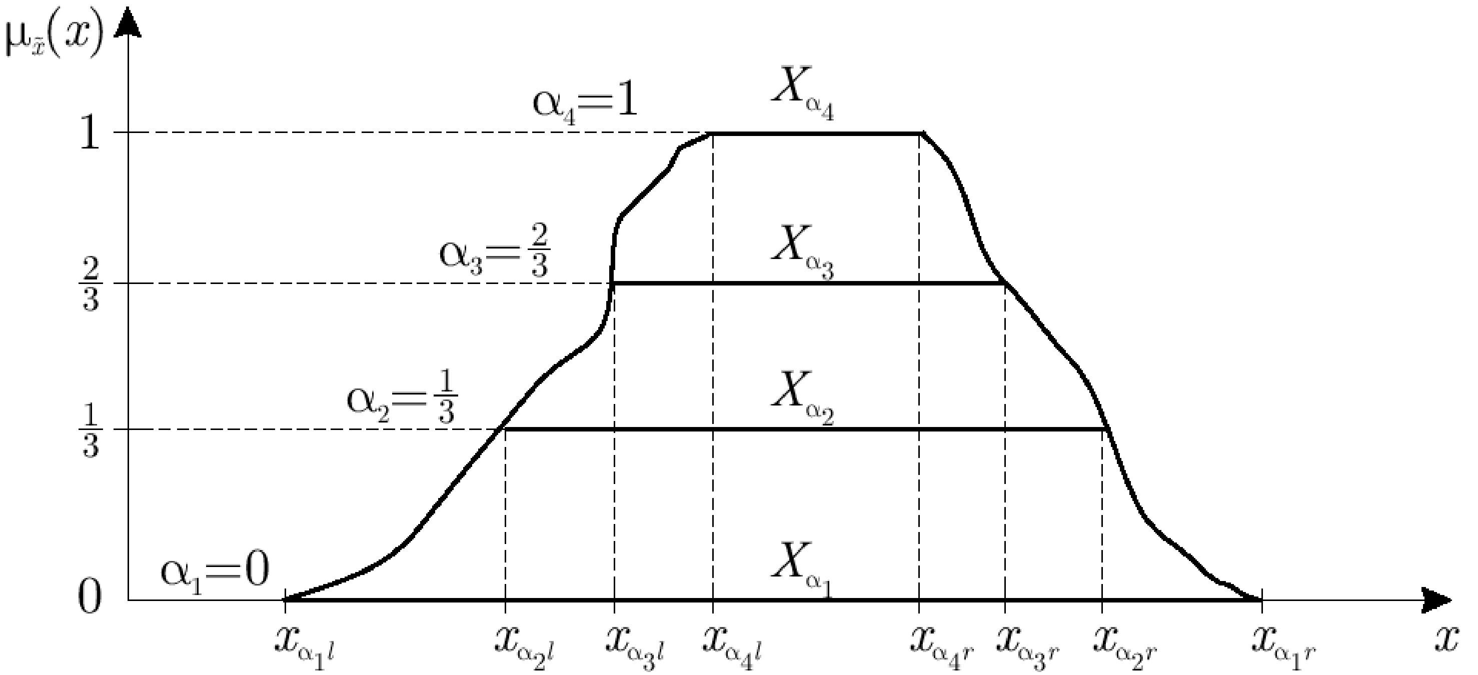
Fuzzy time series



Fuzzy variables

α-discretization

$$\tilde{x} \;=\; \left(X_\alpha = [x_{\alpha l}, x_{\alpha r}] \,\middle|\, \alpha \in [0,1]\right)$$

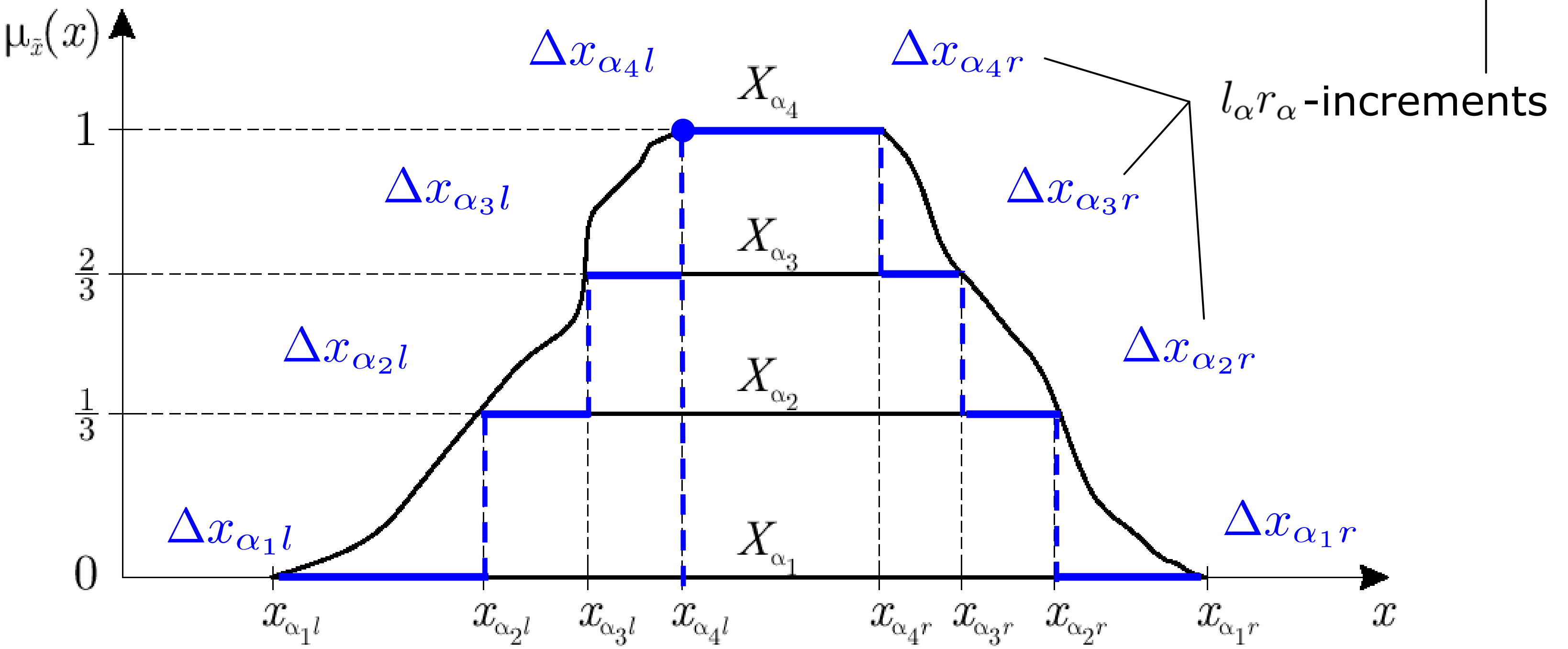


Fuzzy variables

$l_\alpha r_\alpha$-Discretization = discretization with increments

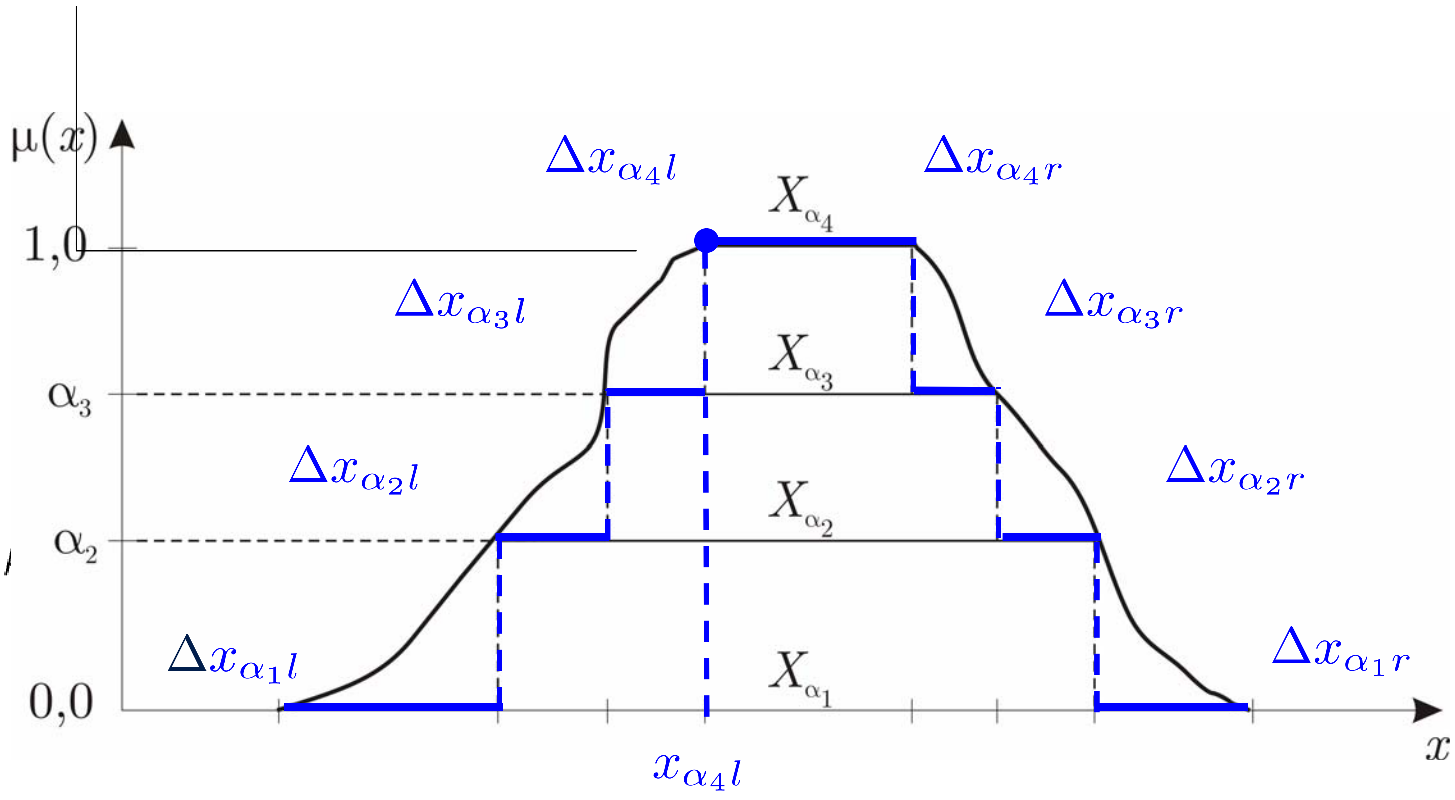
$$X_{\alpha_i} = [x_{\alpha_i l};\, x_{\alpha_i r}] \rightarrow \boxed{\begin{aligned} x_{\alpha_i l} &= x_{\alpha_{i+1} l} - \Delta x_{\alpha_i l} \\ x_{\alpha_i r} &= x_{\alpha_{i+1} r} + \Delta x_{\alpha_i r} \end{aligned}}$$

$$x_{\alpha_n l} = \Delta x_{\alpha_n l}$$
$$x_{\alpha_n r} = x_{\alpha_n l} + \Delta x_{\alpha_n r}$$



Fuzzy variables

$l_\alpha r_\alpha$-Discretization



Fuzzy variables

$$\underline{A} \odot \tilde{x} = \tilde{z}$$

$$\begin{bmatrix} a_{1,1} & a_{2,2} & \cdots & a_{1,2n-1} & a_{1,2n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,2n-1} & a_{2,2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-1,1} & a_{2n-1,2} & \cdots & a_{2n-1,2n-1} & a_{2n-1,2n} \\ a_{2n,1} & a_{2n,2} & \cdots & a_{2n,2n-1} & a_{2n,2n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_{2n-1} \\ \Delta x_{2n} \end{bmatrix} = \begin{bmatrix} \Delta z_1 \\ \Delta z_2 \\ \vdots \\ \Delta z_{2n-1} \\ \Delta z_{2n} \end{bmatrix}$$

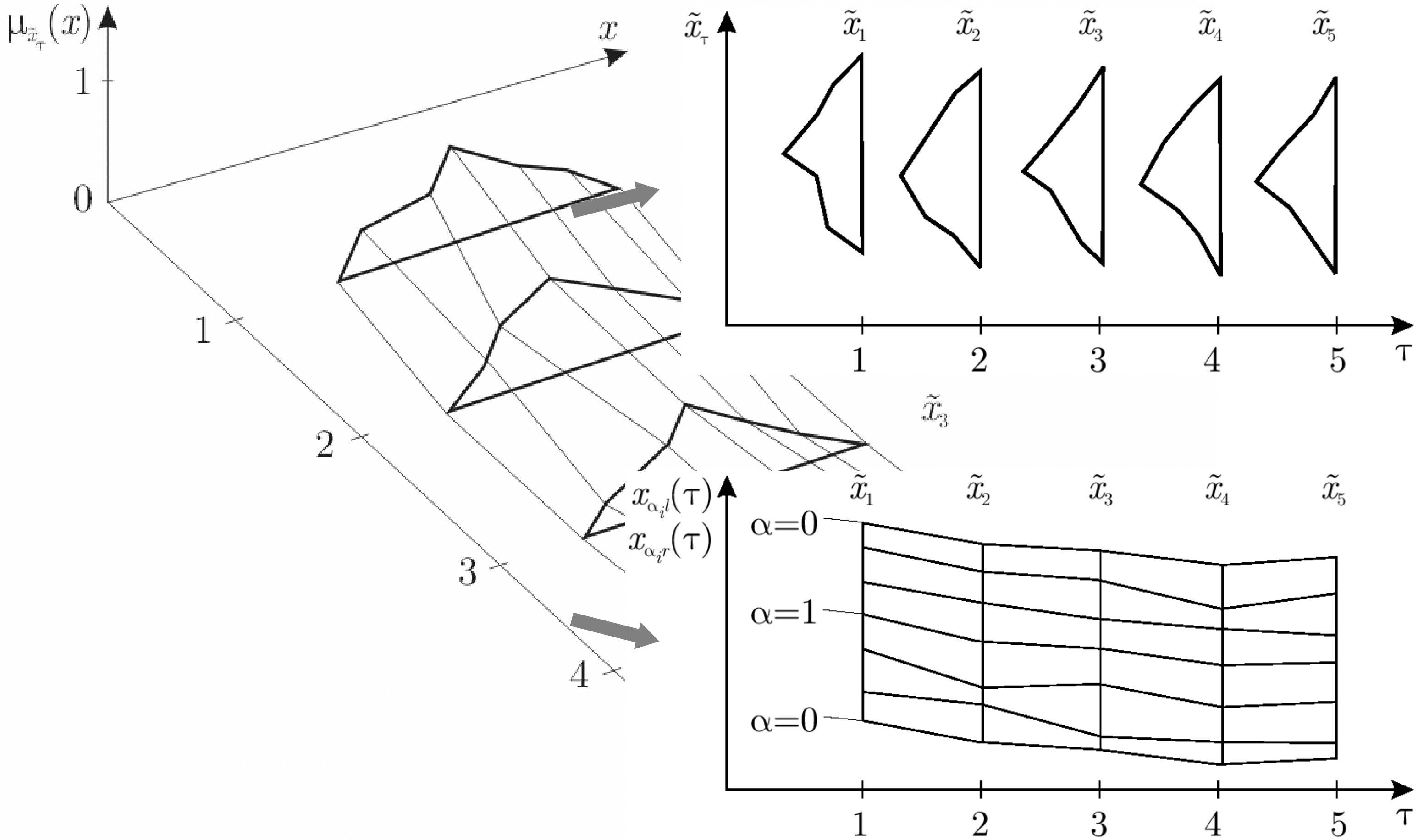
$$\tilde{x} \oplus \tilde{y} = \tilde{z} \longrightarrow \Delta x_i + \Delta y_i = \Delta z_i$$

$$\tilde{x} \ominus \tilde{y} = \tilde{z} \longrightarrow \Delta x_i - \Delta y_i = \Delta z_i$$

$$\boxed{\Delta z_i \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, 2n}$$

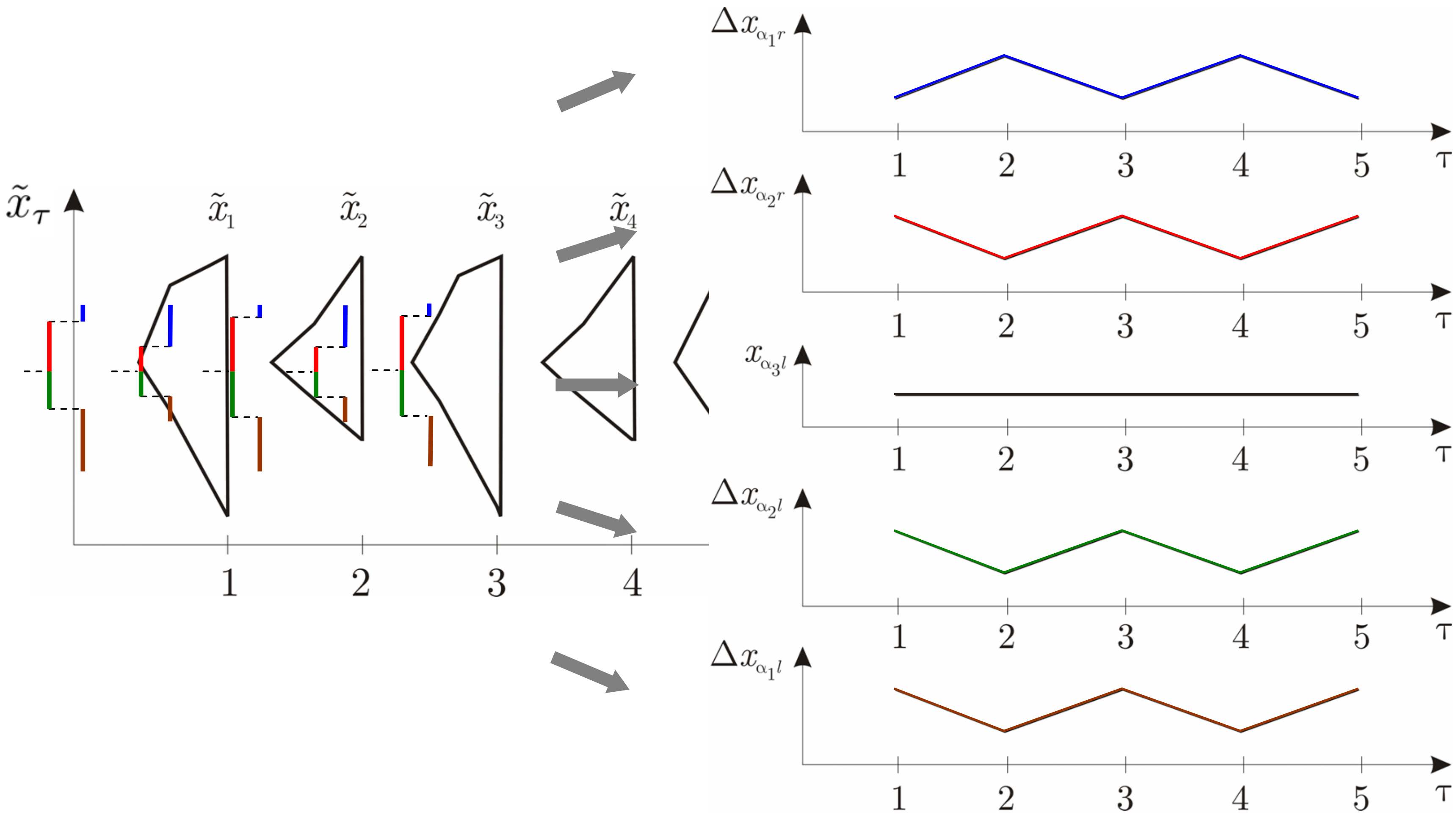
Graphical description of fuzzy time series

→ **Plots**



Graphical description of fuzzy time series

→ **Plots**



Numerical description of fuzzy time series

→ Fuzzy component model

$$\tilde{x}_\tau = \tilde{t}_\tau \oplus \tilde{z}_\tau \oplus \tilde{r}_\tau$$

with: $\tilde{t}_\tau$ … functional value of the fuzzy trend function $\tilde{t}(\tau)$
$\tilde{z}_\tau$ … functional value of the fuzzy cycle function $\tilde{z}(\tau)$
$\tilde{r}_\tau$ … realization of a fuzzy random noise process $\tilde{R}(\tau)$

$$\Delta x_j(\tau) = \Delta t_j(\tau) + \Delta z_j(\tau) + \Delta r_j(\tau) \quad \forall \quad j = 1, 2, ..., 2n$$

$$\left.\begin{array}{r} \Delta x_j(\tau) \geq 0 \\ \Delta t_j(\tau) \geq 0 \\ \Delta z_j(\tau) \geq 0 \\ \Delta r_j(\tau) \geq 0 \end{array}\right\} \quad \forall \quad \tau \in \mathbf{T}, \quad j = 1, 2, ..., n-1, n+1, ..., 2n$$

- **applicable by non-stationary fuzzy time series**

Description of fuzzy time series by empirical parameters

- **applicable by stationary and ergodic fuzzy time series**

→ empirical fuzzy mean value $\tilde{\bar{x}} = \frac{1}{N} \bigoplus_{\tau=1}^{N} \tilde{x}_\tau$

→ empirical $l_\alpha r_\alpha$-covariance function

$$
{}_{lr}\hat{K}_x(\Delta\tau) = \begin{bmatrix} \hat{k}_{1,1}(\Delta\tau) & \hat{k}_{1,2}(\Delta\tau) & \cdots & \hat{k}_{1,2n-1}(\Delta\tau) & \hat{k}_{1,2n}(\Delta\tau) \\ \hat{k}_{2,1}(\Delta\tau) & \hat{k}_{2,2}(\Delta\tau) & \cdots & \hat{k}_{2,2n-1}(\Delta\tau) & \hat{k}_{2,2n}(\Delta\tau) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{k}_{2n-1,1}(\Delta\tau) & \hat{k}_{2n-1,2}(\Delta\tau) & \cdots & \hat{k}_{2n-1,2n-1}(\Delta\tau) & \hat{k}_{2n-1,2n}(\Delta\tau) \\ \hat{k}_{2n,1}(\Delta\tau) & \hat{k}_{2n,2}(\Delta\tau) & \cdots & \hat{k}_{2n,2n-1}(\Delta\tau) & \hat{k}_{2n,2n}(\Delta\tau) \end{bmatrix}
$$

$$
\hat{k}_{i,j}(\Delta\tau) = \frac{1}{N} \sum_{\tau=1}^{N-\Delta\tau} \left[(\Delta x_i(\tau) - \Delta\overline{x}_i) \cdot (\Delta x_j(\tau+\Delta\tau) - \Delta\overline{x}_j) \right]
$$

Modelling of fuzzy time series

Fuzzy time series → realization of a fuzzy random process $\tilde{X}_\tau : \ \Omega \rightarrow \mathbf{F}(\mathbb{R})$

→ family of fuzzy random variables $(\tilde{X}_\tau)_{\tau \in \mathbf{T}}$

$\mathbf{F}(\mathbb{R})$ ... set of all fuzzy variables on $\mathbb{R}$

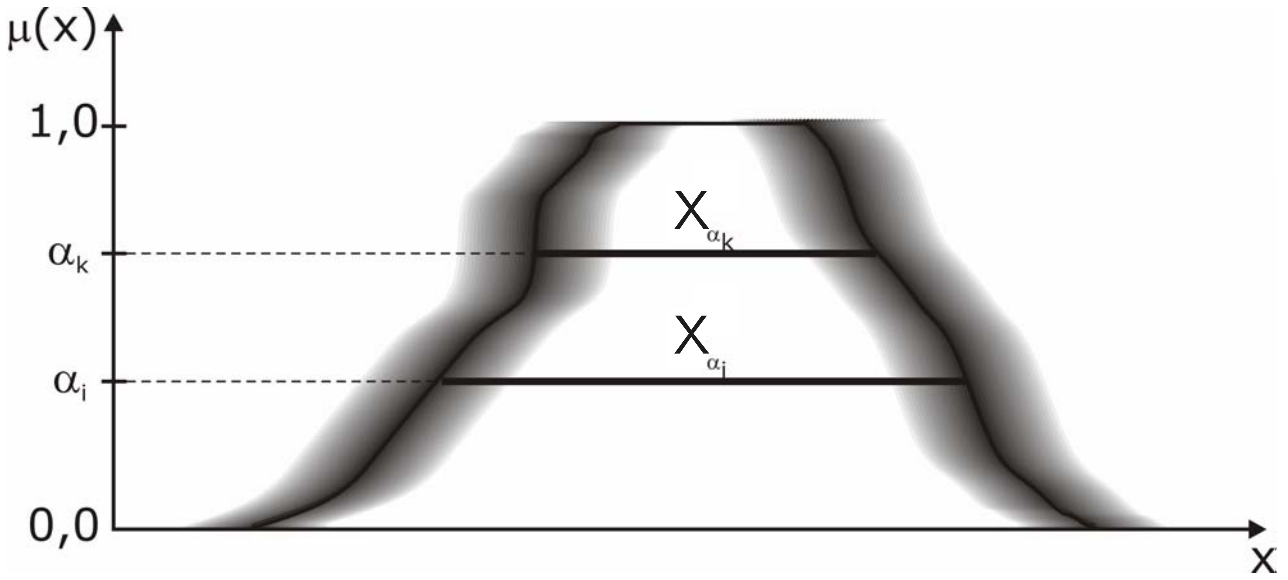
Ω ... space of the random elementary events

→ realizations of a fuzzy random variable are **fuzzy variables**

→ **each realization of a fuzzy random process yields a sequence of fuzzy variables at discrete time points**

Fuzzy random variables

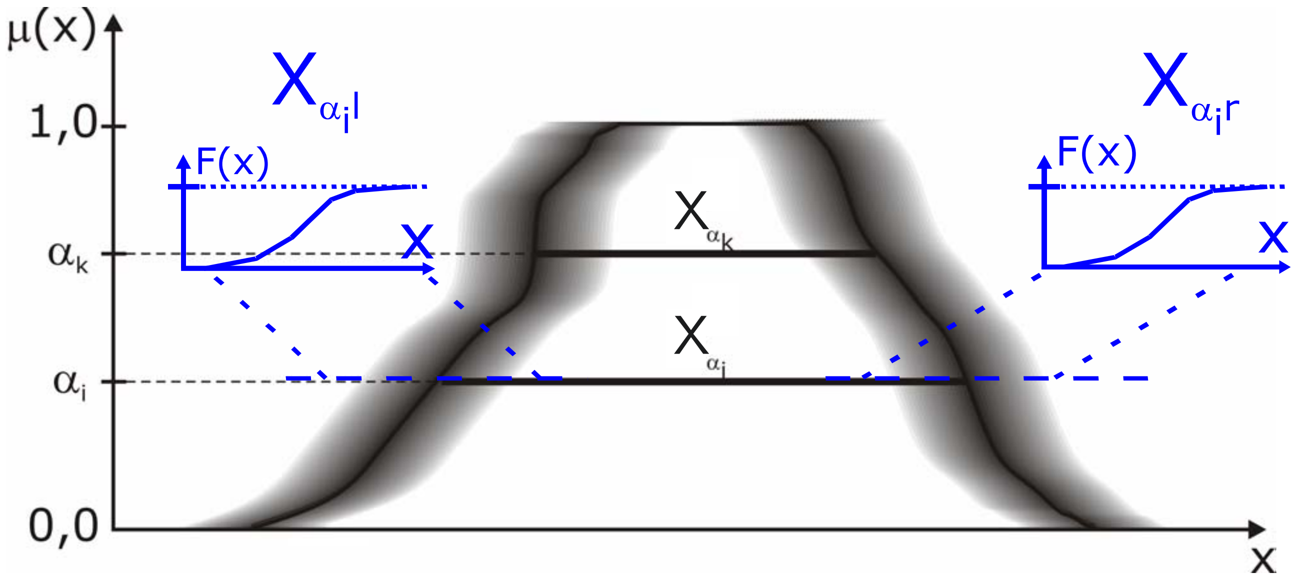
8 realizations



Fuzzy random variables

1. **α-level sets are random sets**

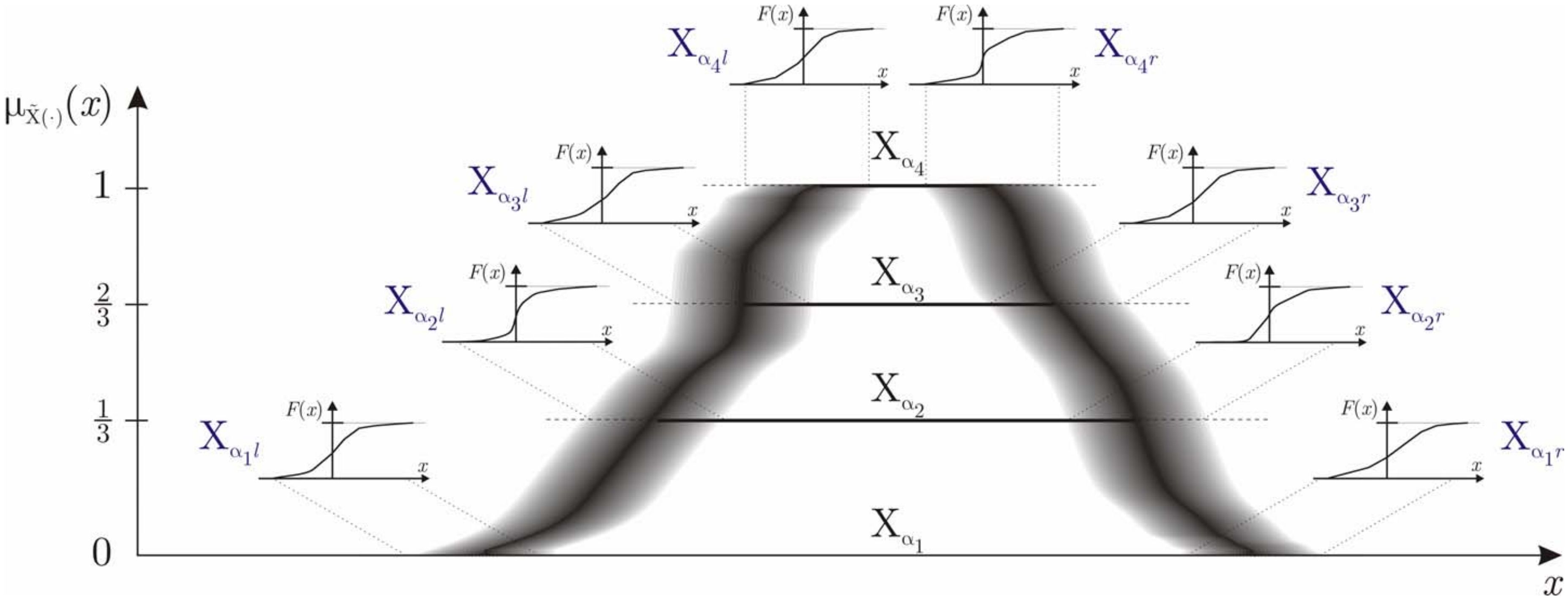
2. $X_{\alpha_i} = [X_{\alpha_i l}\,;\, X_{\alpha_i r}]$ → **interval bounds of the α-level sets are random variables**



Fuzzy random variables

α-Discretization

- $\tilde{X} = (X_\alpha = [X_{\alpha l}, X_{\alpha r}] \mid \alpha \in [0,1])$ →
  - random α-level sets
  - bounds are random variables



Fuzzy random variables

$l_\alpha r_\alpha$-Discretization

- $\mathrm{X}_{\alpha_i} = [\mathrm{X}_{\alpha_i l}; \mathrm{X}_{\alpha_i r}]$ →

$$\boxed{\begin{aligned} \mathrm{X}_{\alpha_i l} &= \mathrm{X}_{\alpha_{i+1} l} - \Delta\mathrm{X}_{\alpha_i l} \\ \mathrm{X}_{\alpha_i r} &= \mathrm{X}_{\alpha_{i+1} r} + \Delta\mathrm{X}_{\alpha_i r} \end{aligned}}$$

$$\mathrm{X}_{\alpha_n l} = \Delta\mathrm{X}_{\alpha_n l}$$

$$\mathrm{X}_{\alpha_n r} = \mathrm{X}_{\alpha_n l} + \Delta\mathrm{X}_{\alpha_n r}$$

$l_\alpha r_\alpha$-Representation

of a fuzzy random variable

$$_{lr}\tilde{\mathrm{X}} = \begin{bmatrix} \Delta\mathrm{X}_{\alpha_1 l} \\ \Delta\mathrm{X}_{\alpha_2 l} \\ \vdots \\ \Delta\mathrm{X}_{\alpha_n l} \\ \Delta\mathrm{X}_{\alpha_n r} \\ \vdots \\ \Delta\mathrm{X}_{\alpha_2 r} \\ \Delta\mathrm{X}_{\alpha_1 r} \end{bmatrix} = \begin{bmatrix} \Delta\mathrm{X}_1 \\ \Delta\mathrm{X}_2 \\ \vdots \\ \Delta\mathrm{X}_n \\ \Delta\mathrm{X}_{n+1} \\ \vdots \\ \Delta\mathrm{X}_{2n-1} \\ \Delta\mathrm{X}_{2n} \end{bmatrix}$$

← **correlated random variables**

Fuzzy random process → $(\tilde{\mathrm{X}}_\tau)_{\tau \in \mathbf{T}}$

- **first and second order moments in $l_\alpha r_\alpha$-representation**

→ fuzzy expected value function

$$E[\tilde{\mathrm{X}}_\tau] = \tilde{m}_{\tilde{\mathrm{X}}_\tau} = \int_0^\infty \cdots \int_{-\infty}^\infty \cdots \int_0^\infty {}_{lr}f_{\tilde{\mathrm{X}}_\tau}(\tilde{x})\, \tilde{x}\; d\Delta x_1 \cdots d\Delta x_n \cdots d\Delta x_{2n}$$

→ $l_\alpha r_\alpha$-covariance function

→ $l_\alpha r_\alpha$-variance

$${}_{lr}Var[\tilde{\mathrm{X}}_\tau] = {}_{lr}\underline{\sigma}^2_{\tilde{\mathrm{X}}_\tau}$$

$${}_{lr}\sigma_i^2(\tau) =$$

for $\tau_a = \tau_b = \tau$

$${}_{lr}K_{\tilde{\mathrm{X}}_\tau}(\tau_a, \tau_b) = \begin{bmatrix} & k_{1,2}(\tau_a,\tau_b) & \cdots & k_{1,2n-1}(\tau_a,\tau_b) & k_{1,2n}(\tau_a,\tau_b) \\ k_{2,1}(\tau_a,\tau_b) & & \cdots & k_{2,2n-1}(\tau_a,\tau_b) & k_{2,2n}(\tau_a,\tau_b) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & \cdots & k_{2n-1,2n-1}(\tau_a,\tau_b) & k_{2n-1,2n}(\tau_a,\tau_b) \\ & & & k_{2n,2n-1}(\tau_a,\tau_b) & k_{2n,2n}(\tau_a,\tau_b) \end{bmatrix}$$

$$k_{i,j}(\tau_a,\tau_b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\Delta x_i - \Delta m_i(\tau_a))\,(\Delta x_j - \Delta m_j(\tau_b)) \ldots$$

$$\ldots f\,(\Delta x_i, \Delta x_j, \tau_a, \tau_b)\; d\Delta x_i d\Delta x_j$$

Fuzzy-white-noise-process

$\rightarrow (\tilde{\mathcal{E}}_\tau)_{\tau \in \mathbf{T}}$

stationary and ergodic process

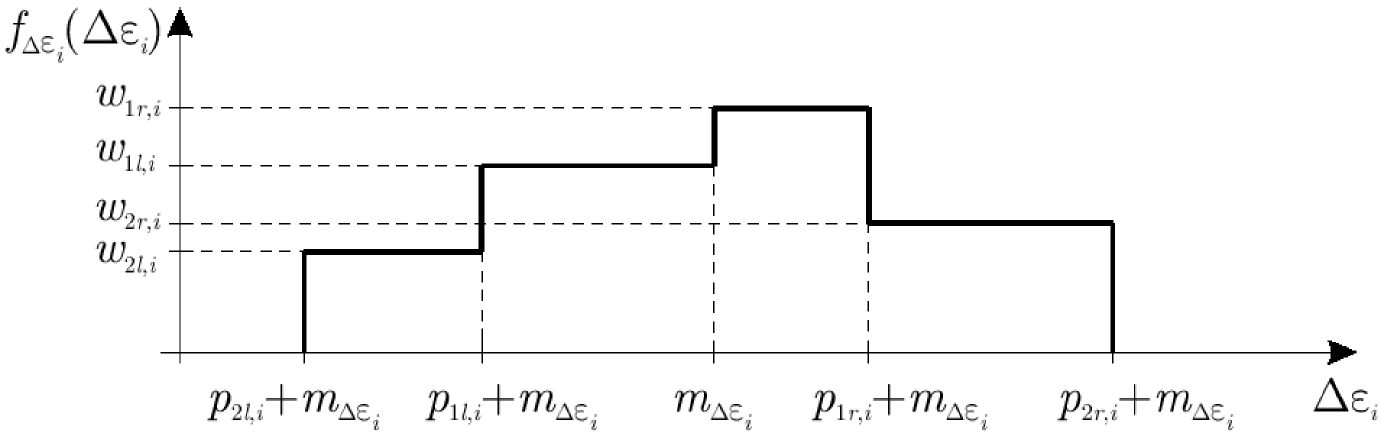
$$E[\tilde{\mathcal{E}}_\tau] \;=\; \tilde{m}_{\tilde{\mathcal{E}}} = \text{constant} \quad \forall\, \tau \in \mathbf{T}$$

$$_{lr}Var[\tilde{\mathcal{E}}_\tau] \;=\; {}_{lr}\underline{\sigma}^2_{\tilde{\mathcal{E}}} = \text{constant} \quad \forall\, \tau \in \mathbf{T}$$

$$_{lr}K_{\tilde{\mathcal{E}}}(\Delta\tau) \;=\; \begin{cases} {}_{lr}K_{\tilde{\mathcal{E}}}(0) & \text{for} \quad \Delta\tau = 0 \\ \underline{0} & \text{for} \quad \Delta\tau \neq 0 \end{cases}$$

increments are dependent
increments are independent

→ probability density function of a random increment $\Delta\varepsilon_i$



Fuzzy-ARMA-process

$\rightarrow \quad (\tilde{X}_\tau)_{\tau \in \mathbf{T}}$

$$\tilde{X}_\tau = \underbrace{\underline{A}_1 \odot \tilde{X}_{\tau-1} \oplus \ldots \oplus \underline{A}_p \odot \tilde{X}_{\tau-p} \oplus \tilde{\mathcal{E}}_\tau}_{\text{Fuzzy-AR-process}} \underbrace{\ominus \underline{B}_1 \odot \tilde{\mathcal{E}}_{\tau-1} \ominus \ldots \ominus \underline{B}_q \odot \tilde{\mathcal{E}}_{\tau-q}}_{\text{Fuzzy-MA-process}}$$

with: $\underline{A}_j$, $\underline{B}_j$ … real valued [2n,2n] parameter matrices

$\tilde{\mathcal{E}}_\tau$ … fuzzy random variable of a fuzzy-white-noise-process

Forecasting presupposes the estimation of the parameter matrices.

Estimation of the parameter $\{\underline{A}_1,\ldots,\underline{A}_p, \underline{B}_1,\ldots,\underline{B}_q\} = \underline{P}$

3 strategies for parameter estimation:

Idea 1: **Minimization of the differences between empirical parameters and model parameters**

$$\sum_{j=1}^{2n} \left(\Delta\bar{x}_j - \Delta m_j(\underline{P})\right)^2 + \sum_{\Delta\tau=-\infty}^{\infty} \sum_{k,l=1}^{2n} \left(\hat{r}_{k,l}(\Delta\tau) - r_{k,l}(\Delta\tau, \underline{P})\right)^2 \stackrel{!}{=} \min$$

applicable for stationary and ergodic fuzzy time series

Estimation of the parameter {$\underline{A}_1,\dots,\underline{A}_p$, $\underline{B}_1,\dots,\underline{B}_q$} = $\underline{P}$

Idea 2: **Minimization of the average distance between measured fuzzy data and optimal 1-step-forecasting**

$$\overline{d}_F(\underline{P}) = \frac{1}{N-p}\sum_{\tau=p+1}^{N} \boxed{d_F}\left(\tilde{x}_\tau;\, \mathring{\tilde{x}}_\tau(\underline{P})\right) \overset{!}{=} \min$$

distance function

$$\boxed{d_F}\left(\tilde{x}_\tau;\, \hat{\tilde{x}}_\tau(\underline{P})\right) = \int_0^1 \boxed{d_H}\left(X_\alpha(\tilde{x}_\tau);\, X_\alpha(\hat{\tilde{x}}_\tau(\underline{P}))\right) d\alpha$$

HAUSDORFF distance

$$\boxed{d_H}(A;B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d_E(a;b)\,;\, \sup_{b\in B}\inf_{a\in A} d_E(a;b)\right\}$$

applicable for non-stationary fuzzy time series

Estimation of the parameter $\{\underline{A}_1,\ldots,\underline{A}_p, \underline{B}_1,\ldots,\underline{B}_q\} = \underline{P}$

Idea 3: **Minimization of the square error between measured fuzzy data and optimal 1-step-forecasting**

$$E = \frac{1}{2}\sum_{\tau=1+p}^{N}\sum_{i=1}^{2n}\left(\Delta x_i(\tau) - \Delta\mathring{x}_i(\tau, \underline{P})\right)^2 \overset{!}{=} \min$$

incremental improvement of the parameter

$$\underline{A}_r(\text{new}) = \underline{A}_r(\text{old}) + \Delta\underline{A}_r \qquad r = 1, 2, ..., p$$

$$\underline{B}_s(\text{new}) = \underline{B}_s(\text{old}) + \Delta\underline{B}_s \qquad s = 1, 2, ..., q$$

e.g.:

$$\Delta\underline{A}_r = -\eta\frac{\partial E}{\partial \underline{A}_r} = -\eta\begin{bmatrix} \frac{\partial E}{\partial a_{1,1}(r)} & \cdots & \frac{\partial E}{\partial a_{1,2n}(r)} \\ \vdots & \ddots & \vdots \\ \frac{\partial E}{\partial a_{2n,1}(r)} & \cdots & \frac{\partial E}{\partial a_{2n,2n}(r)} \end{bmatrix}$$

applicable for non-stationary fuzzy time series

Forecasting strategies

- future values of a fuzzy time series are realizations of a fuzzy random forecast process

Fuzzy random forecast process
= family of conditional random variables

conditional fuzzy random variable

$(\vec{\tilde{X}}_{\tau})_{\tau \in \mathbf{T}}$ with $\vec{\tilde{X}}_{N+h}(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N)$

realizations of the forecast process are future values of the fuzzy time series

Forecasting strategies

1. Optimal forecasting

$$\mathring{\tilde{x}}_{N+h}(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N) = E[\tilde{\mathrm{X}}_{N+h} \,|\, \tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N] = E[\vec{\vec{\tilde{\mathrm{X}}}}_{N+h}]$$

optimal forecasted value

conditional fuzzy expected value of $\tilde{\mathrm{X}}_{N+h}$

- **optimal 1-step-forecasting** (Fuzzy-ARMA[p,q]-process)

$$\mathring{\tilde{x}}_{N+1} \;=\; \underline{A}_1 \odot \tilde{x}_N \oplus ... \oplus \underline{A}_p \odot \tilde{x}_{N+1-p} \oplus E[\tilde{\mathcal{E}}_\tau] \ominus \underline{B}_1 \odot \tilde{\varepsilon}_N \ominus ... \ominus \underline{B}_q \odot \tilde{\varepsilon}_{N+1-q}$$

- **optimal h-step-forecasting** (Fuzzy-ARMA[p,q]-process)

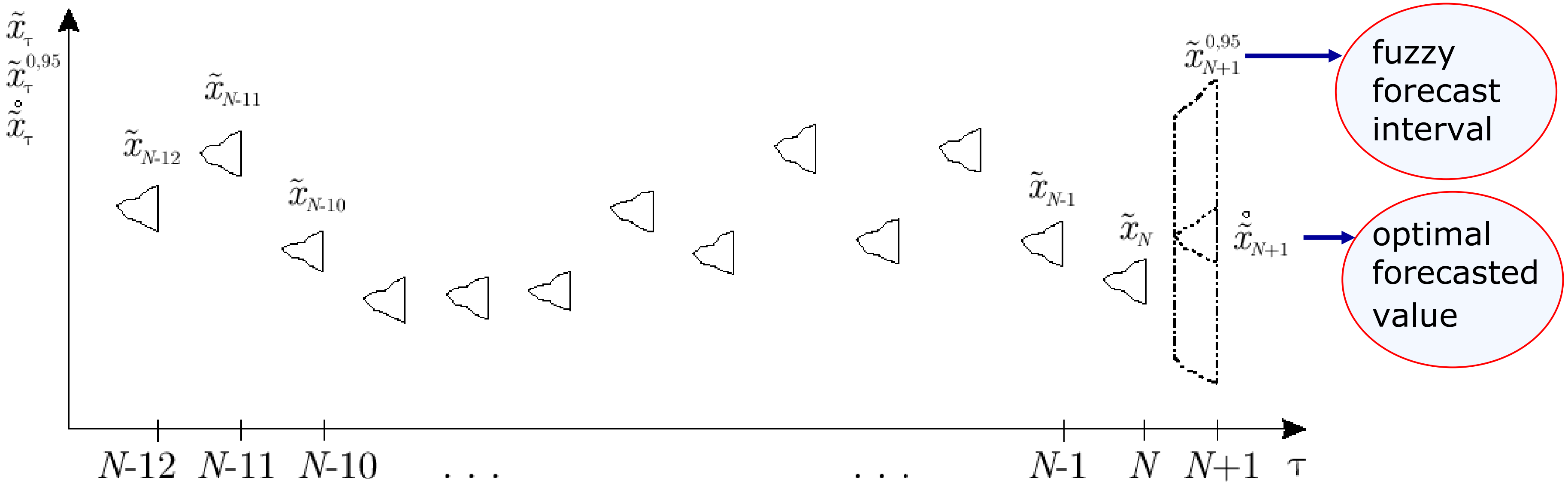
$$\mathring{\tilde{x}}_{N+h} \;=\; \underline{A}_1 \odot \tilde{x}_{N+h-1} \oplus ... \oplus \underline{A}_p \odot \tilde{x}_{N+h-p} \oplus E[\tilde{\mathcal{E}}_\tau] \ominus \underline{B}_1 \odot \hat{\tilde{\varepsilon}}_{N+h-1} \ominus ... \ominus \underline{B}_q \odot \hat{\tilde{\varepsilon}}_{N+h-q}$$

with $\tilde{x}_{N+u} = \begin{cases} \tilde{x}_{N+u} & \text{for} \quad u \leq 0 \\ \mathring{\tilde{x}}_{N+u} & \text{for} \quad u > 0 \end{cases}$ and $\tilde{\varepsilon}_{N+u} = \begin{cases} \tilde{\varepsilon}_{N+u} & \text{for} \quad u \leq 0 \\ E[\tilde{\mathcal{E}}_\tau] & \text{for} \quad u > 0 \end{cases}$

Forecasting strategies

2. Fuzzy forecast intervals

- A fuzzy forecast interval $\tilde{x}^{\kappa}_{N+h}$ includes realizations $\vec{\tilde{x}}_{N+h}$ with probability κ



- **s future realizations** are simulated by Monte Carlo Simulation

Forecasting strategies

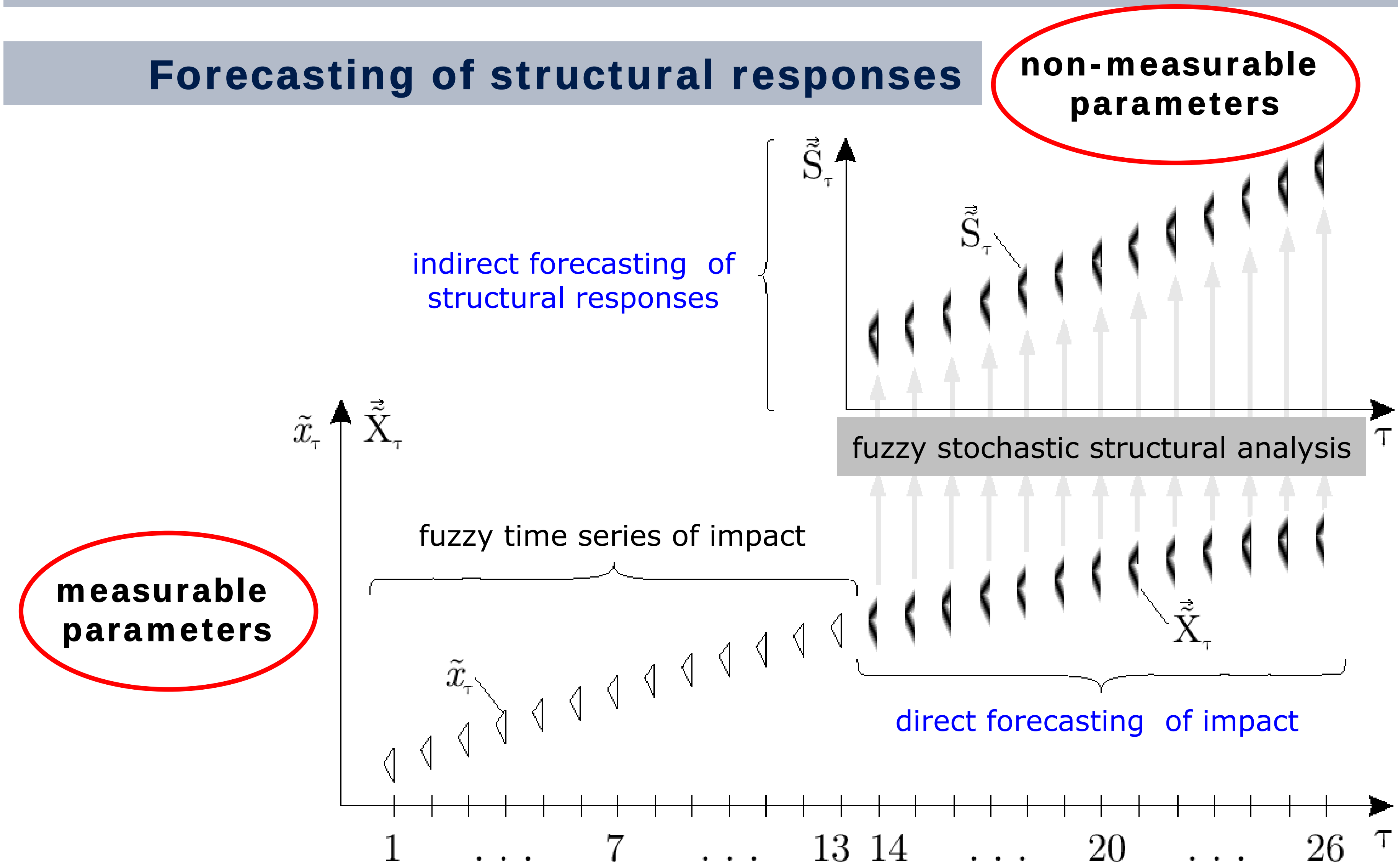
3. Fuzzy random forecasting

objective: estimation of the fuzzy random variable $\tilde{\vec{X}}_{N+h}$

- Monte Carlo simulation of s fuzzy variables $\tilde{\vec{x}}_{N+h}$
- statistical evaluation of the simulated fuzzy variables $\tilde{\vec{x}}_{N+h}$

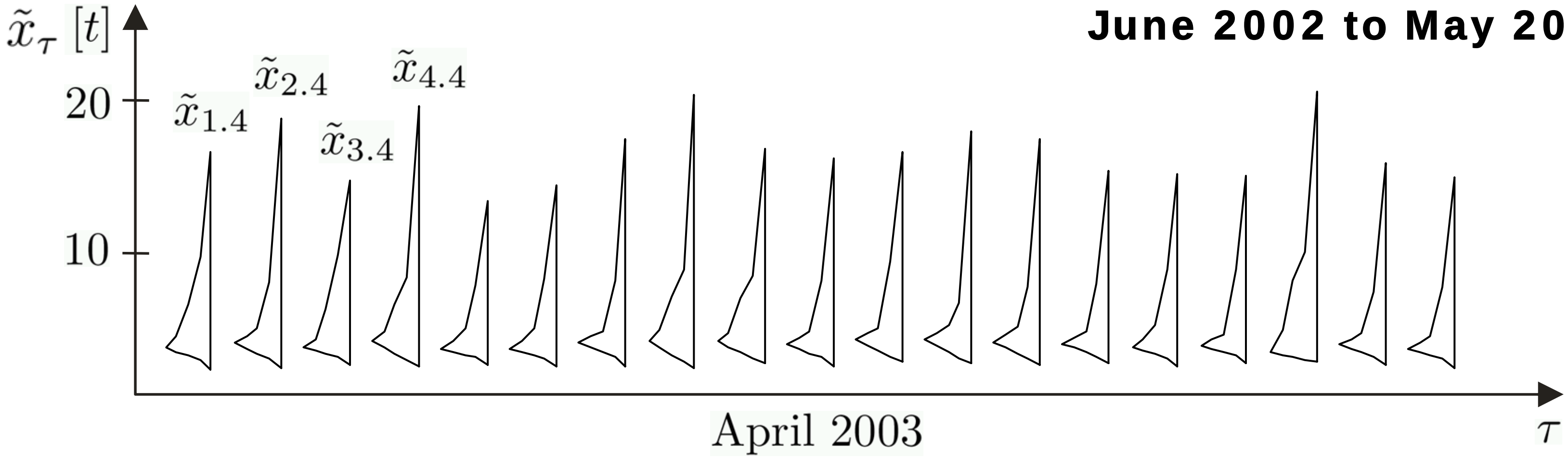
e.g.
- fuzzy probability distribution function ${}_{lr}F_{\tilde{\vec{X}}}(\tilde{x})$
- fuzzy expected value $E[\tilde{\vec{X}}_{N+h}]$
- $l_\alpha r_\alpha$-covariance function ${}_{lr}K_{\tilde{\vec{X}}}(\tau_a, \tau_b)$

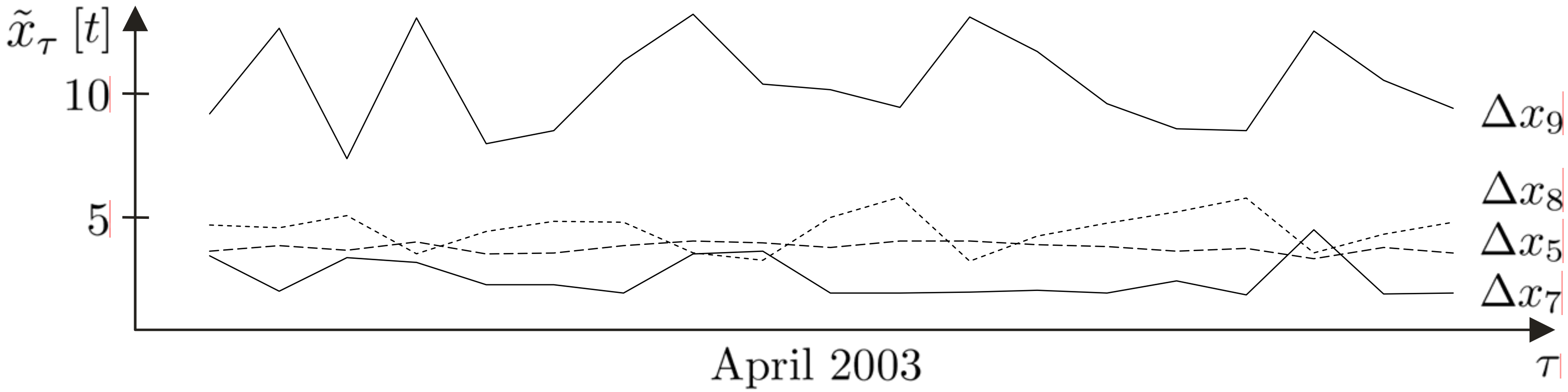
Forecasting of structural responses



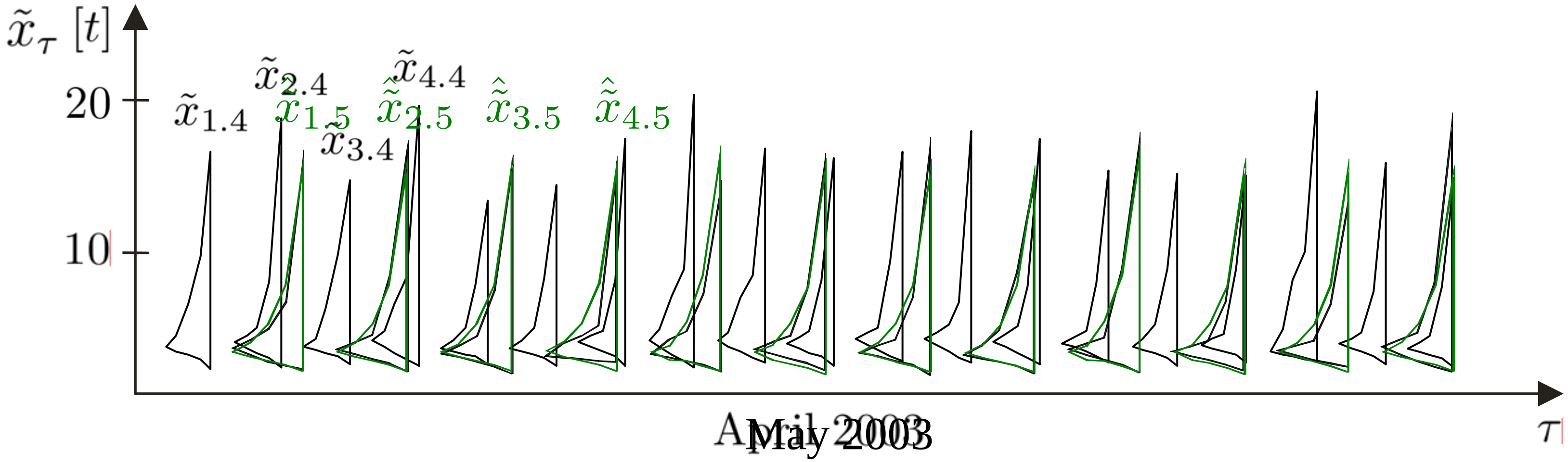
measured data of heavy goods vehicle traffic

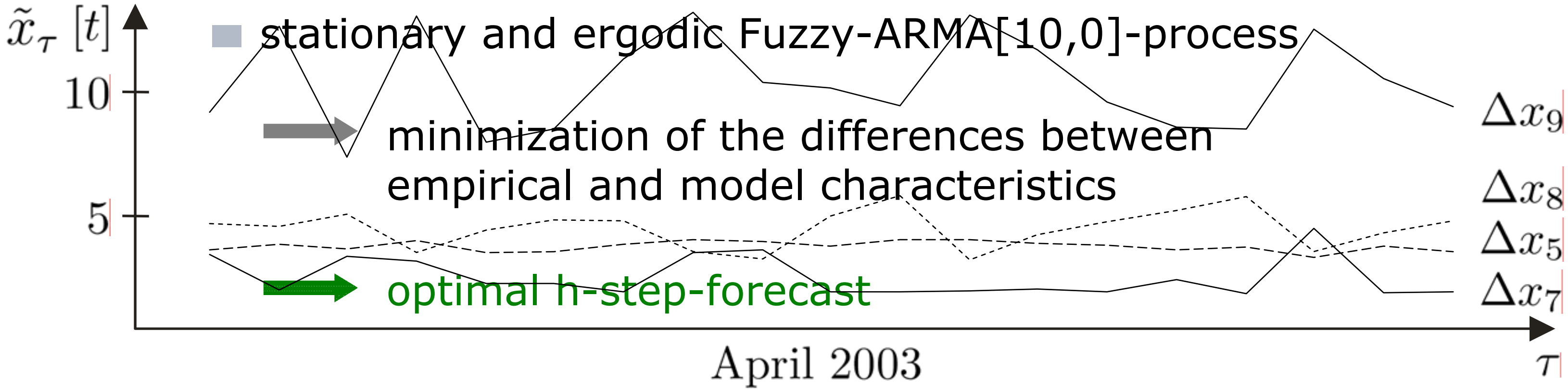
June 2002 to May 2003





Measured data of heavy goods vehicle traffic





- stationary and ergodic Fuzzy-ARMA[10,0]-process

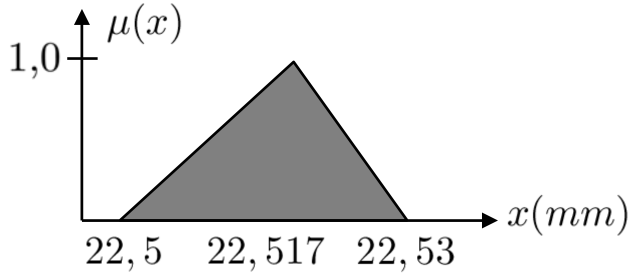
→ minimization of the differences between empirical and model characteristics

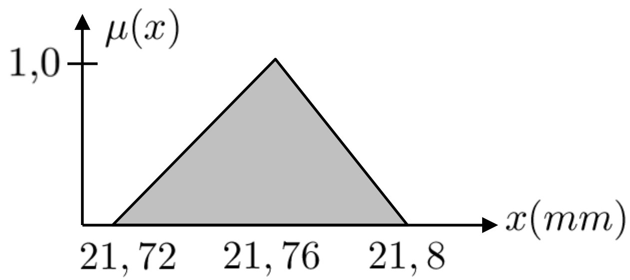
→ optimal h-step-forecast

Time series with measured settlements

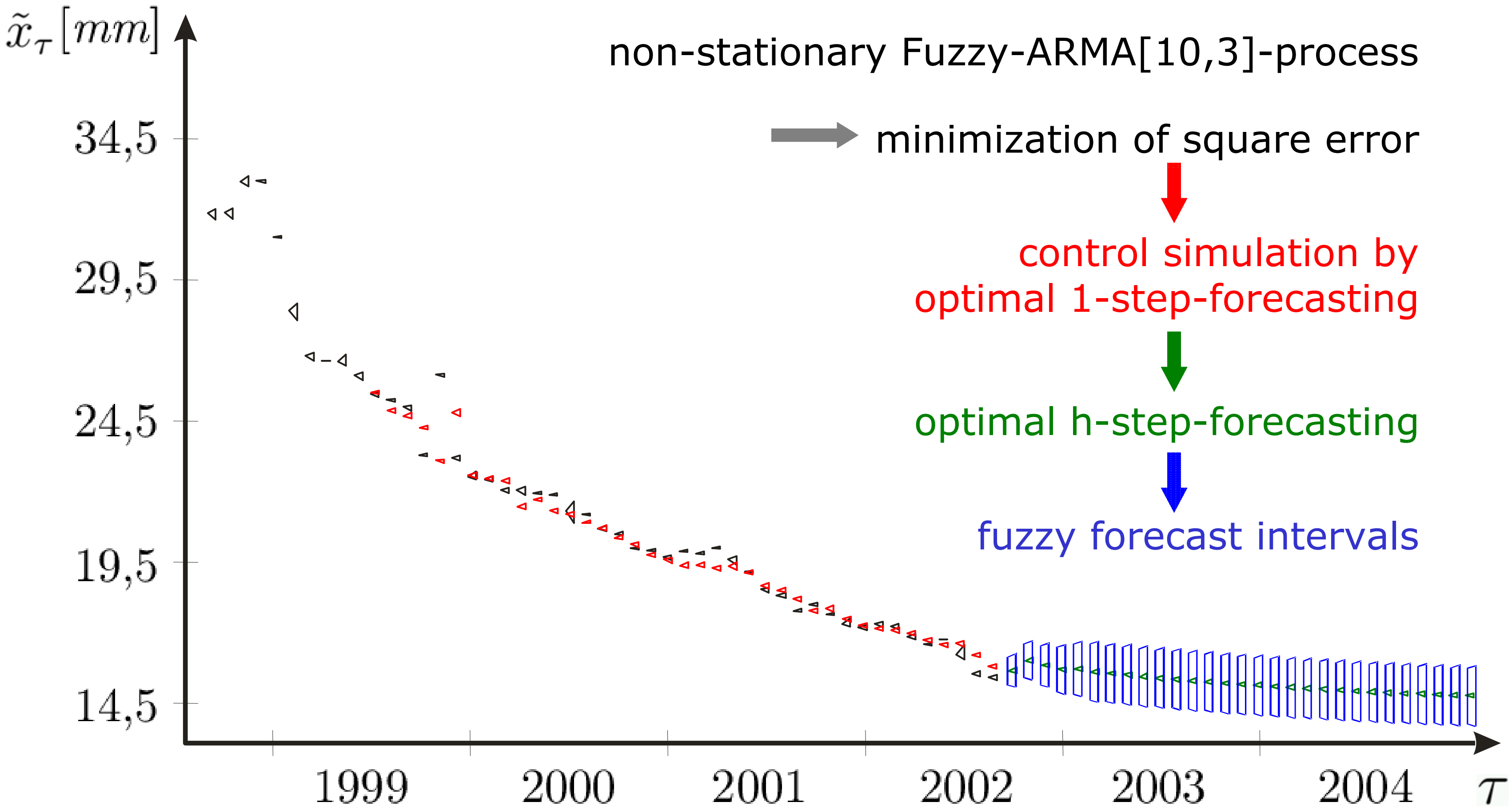
(over 4 years)

date	1st measurement [mm]	2nd measurement [mm]	3rd measurement [mm]	mean value [mm]
⋮	⋮	⋮	⋮	⋮
30.05.2000	22,51	22,50	22,52	22,510
27.06.2000	22,50	22,52	22,53	22,517
27.07.2000	22,40	22,40	22,41	22,403
30.08.2000	22,35	22,36	22,35	22,353
27.09.2000	21,72	21,80	21,77	21,763
⋮	⋮	⋮	⋮	⋮



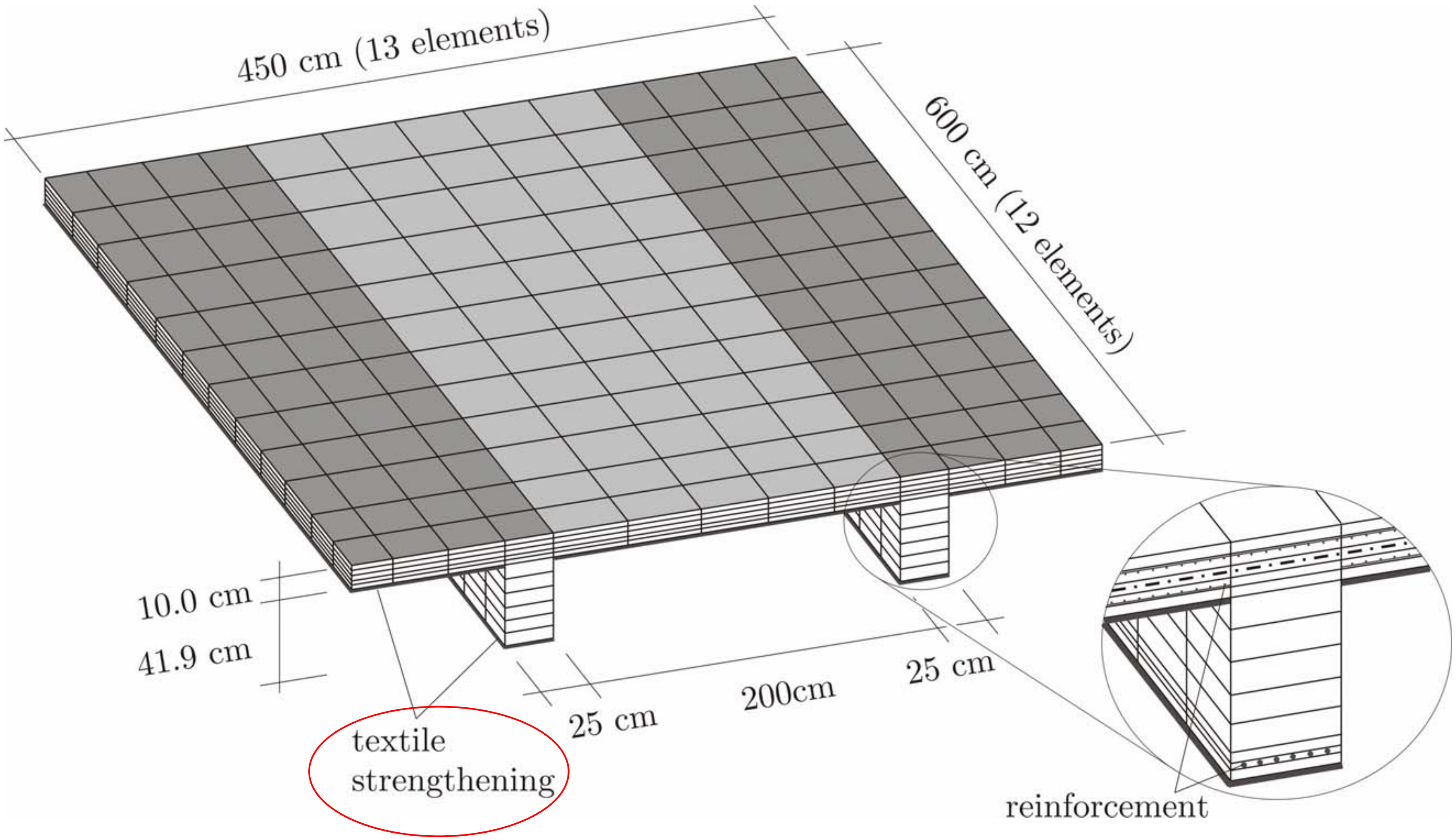


Measured data of an extensometer



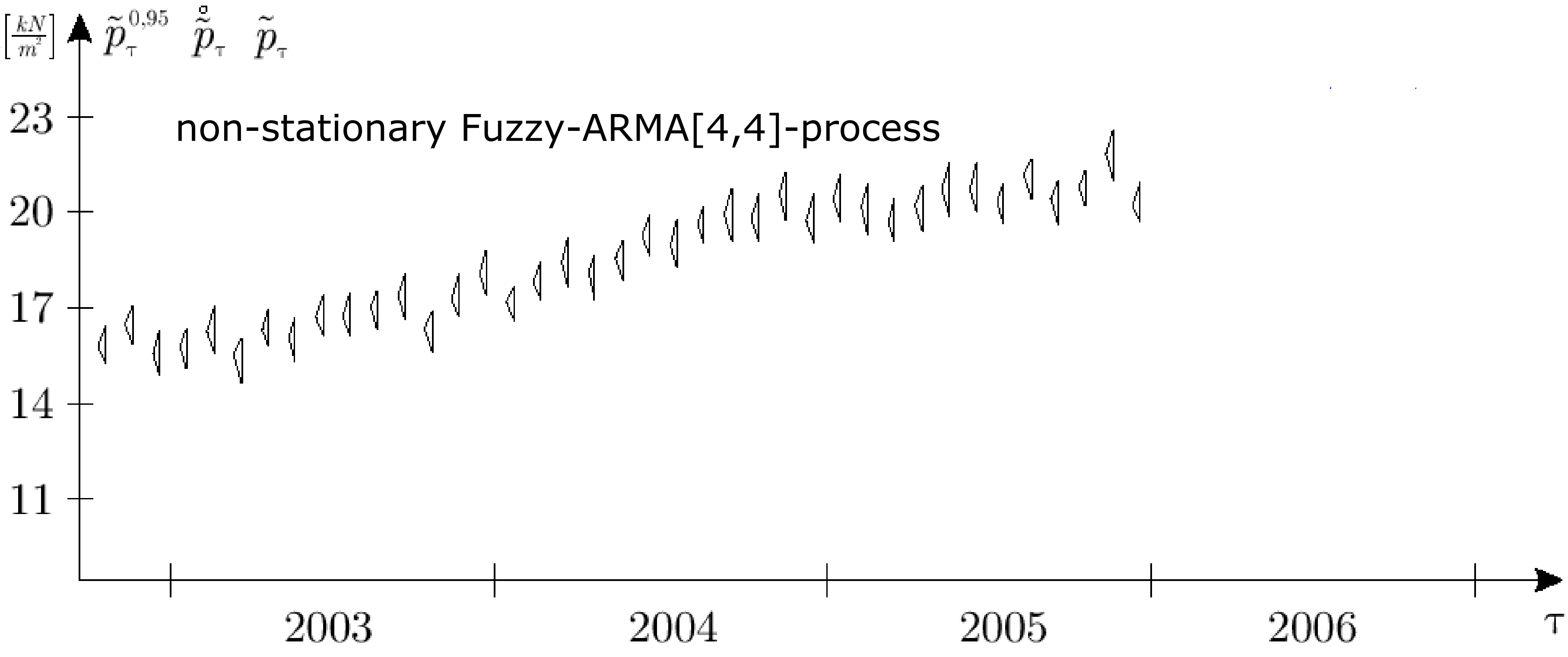
Damage of a T-beam plate

indirect forecasting



Damage of a T-beam plate

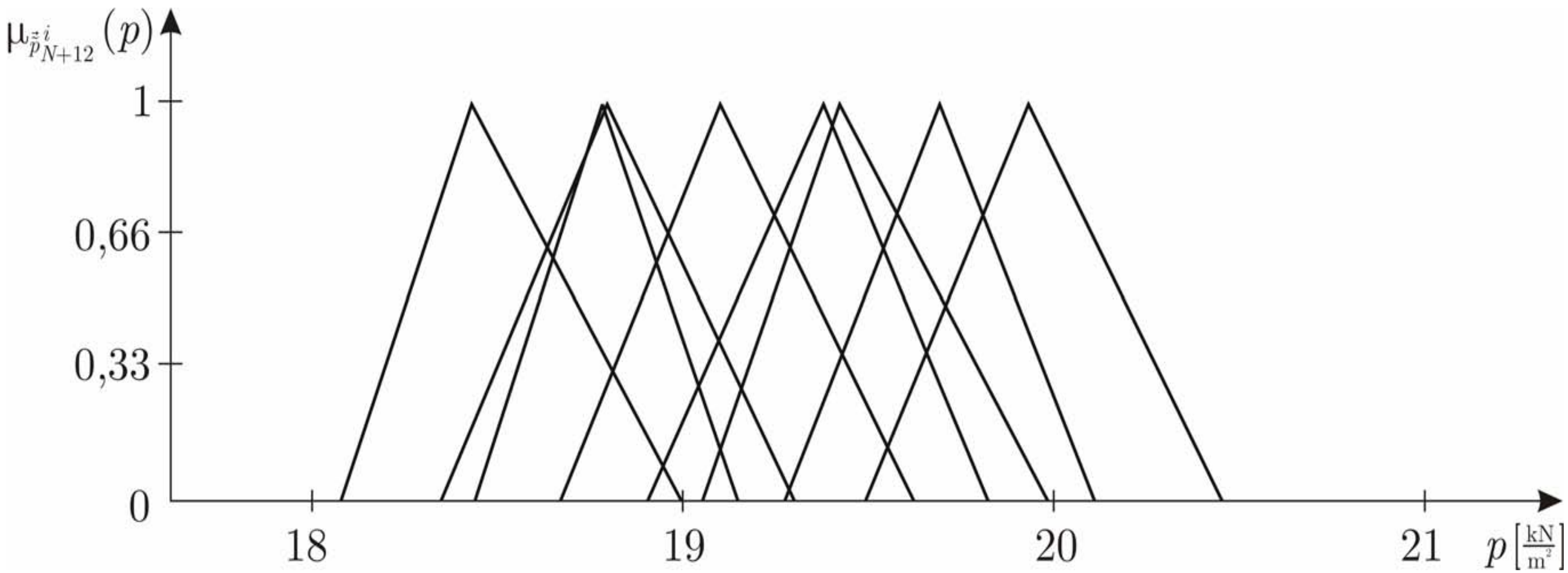
Time series of live laods – monthly goods – in a storehouse



Damage of a T-beam plate

Fuzzy random forecasting

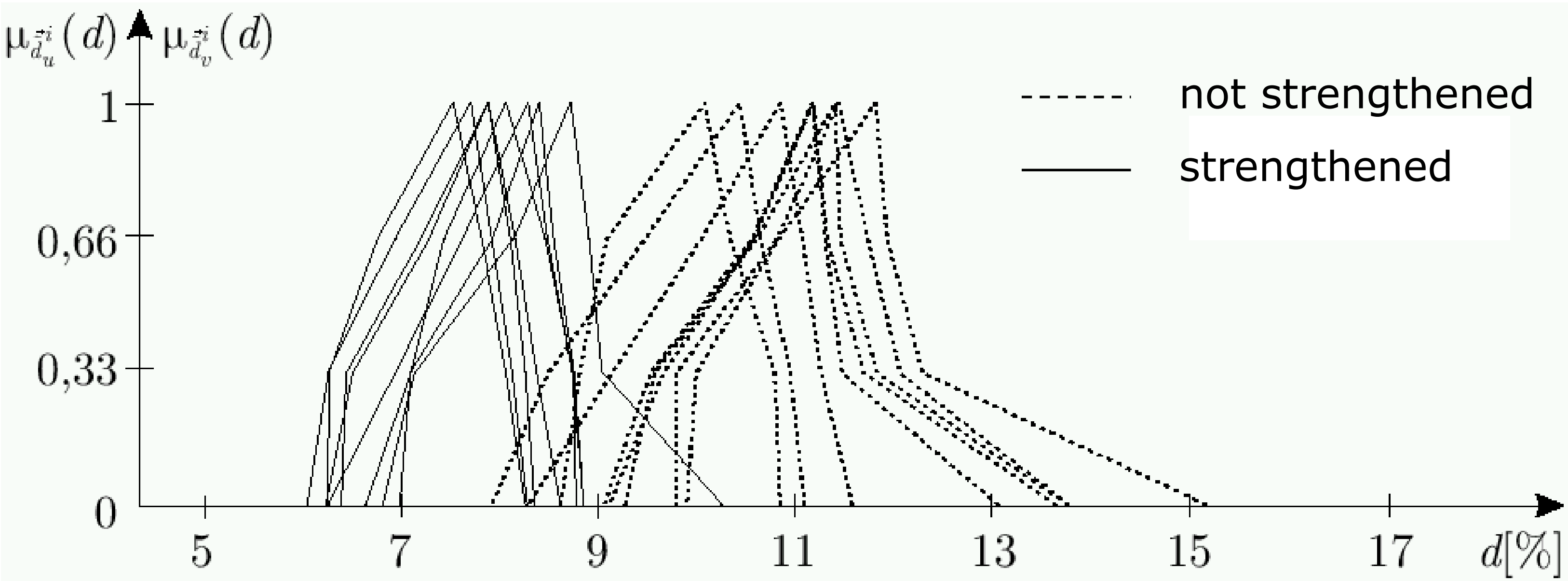
- Monte Carlo Simulation of 100 future realizations
- 8 realizations $\tilde{\vec{p}}_{N+12}$ at time point $\tau = N + 12$



Damage of a T-beam plate

indirect forecasting

- Fuzzy damage indicator $\tilde{D}_{\underline{K}} = 1 - \dfrac{\det\left[\underline{\tilde{K}}_T(\tau, \underline{\tilde{v}}, \tilde{s})\right]}{\det\left[\underline{\tilde{K}}_T(\tau_0, \underline{\tilde{v}}_0, \tilde{s}_0)\right]}$



Conclusions

- Time series with fuzzy data can be modeled as realizations of fuzzy random processes

- New $l_\alpha r_\alpha$-representation of fuzzy random variables enables the modeling of fuzzy time series as realizations of fuzzy random processes

- Fuzzy-ARMA-processes allow the simulation and forecasting of fuzzy time series

Thank you!