TECHNISCHE UNIVERSITÄT DRESDEN

Institute for Statics und Dynamics of Structures

Fuzzy Stochastic Finite Element Method

-

FSFEM

Bernd Möller

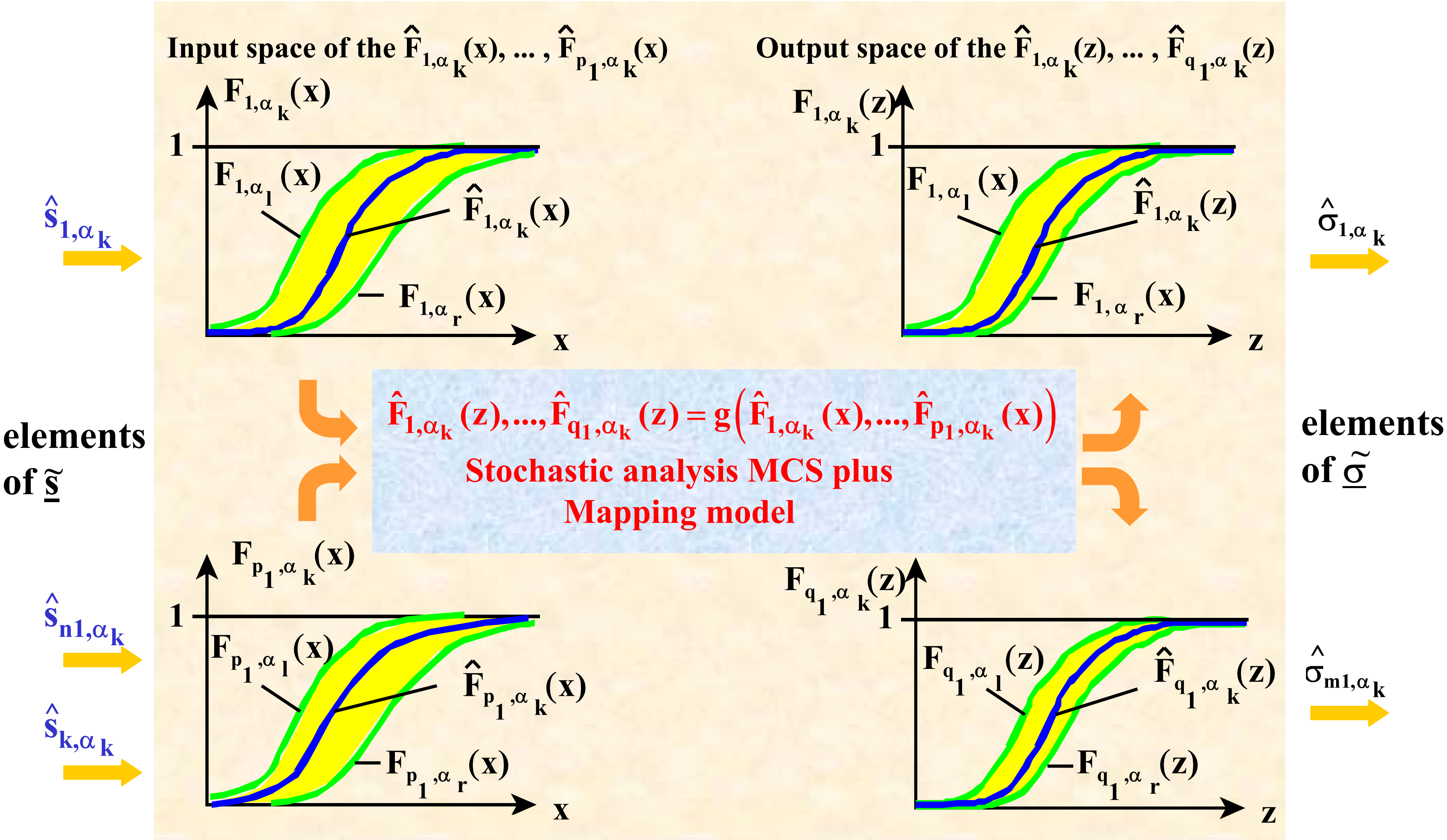
Fuzzy Stochastic Analysis

from fuzzy stochastic sampling to fuzzy stochastic analysis

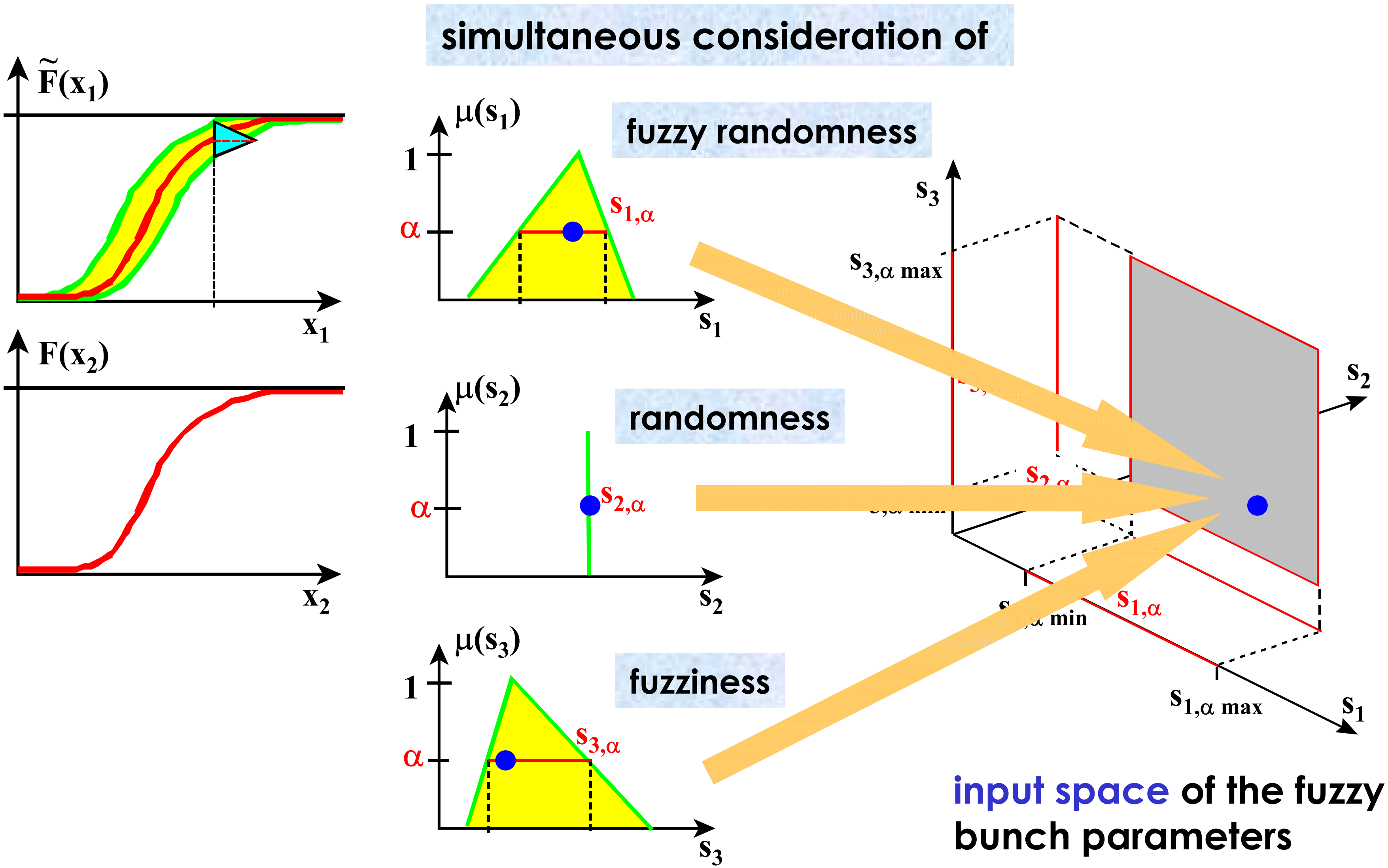
Fuzzy Stochastic Analysis

Uncertain input data	Mapping	Uncertain result data
Fuzzy random variables $\tilde{X}$ Fuzzy Random functions $X(\tilde{s}, \underline{t})$	$\underline{\tilde{X}} \xrightarrow{\sim} \underline{\tilde{Z}}$ α-discretization yields: original X_j of $\tilde{X}$ with assigned probability distribution function $F_j(x)$ Stochastic analysis with the X_j and MCS plus algorithm for structural analysis	Combining the results $F_j(z)$ to the fuzzy probability distribution function $F(\tilde{\sigma},z)$ of the fuzzy random result $\tilde{Z}$ (empirical) probability distribution function $F_j(z)$ for structural response parameter Z_j (at regarded α-level)

Fuzzy Stochastic Analysis: Mapping Operator



Fuzzy Stochastic Analysis



input space of the fuzzy bunch parameters

Fuzzy Stochastic Finite Element Method
–
FSFEM

**from fuzzy stochastic sampling to
fuzzy stochastic finite element method**

FSFEM:

Fuzzy random input variables → $\mathbf{F}: \underline{\tilde{\mathbf{X}}}(\underline{\mathbf{t}}) \rightarrow \underline{\tilde{\mathbf{Z}}}(\underline{\mathbf{t}}) \quad \left(\underline{\mathbf{t}} = \left\{\underline{\theta}, \tau, \underline{\varphi}\right\}\right)$ → **Fuzzy random result variables**

Differentional equation of motion

$$\tilde{\mathbf{m}}_{\mathbf{t}}(.) \cdot \ddot{\tilde{\mathbf{v}}} + \tilde{\mathbf{d}}_{\mathbf{t}}(.) \cdot \dot{\tilde{\mathbf{v}}} + \tilde{\mathbf{k}}_{\mathbf{t}}(.) \cdot \tilde{\mathbf{v}} = \tilde{\mathbf{f}}_{\mathbf{t}}(.) \quad \forall \ \underline{\mathbf{t}} \in \underline{\mathbf{T}}$$

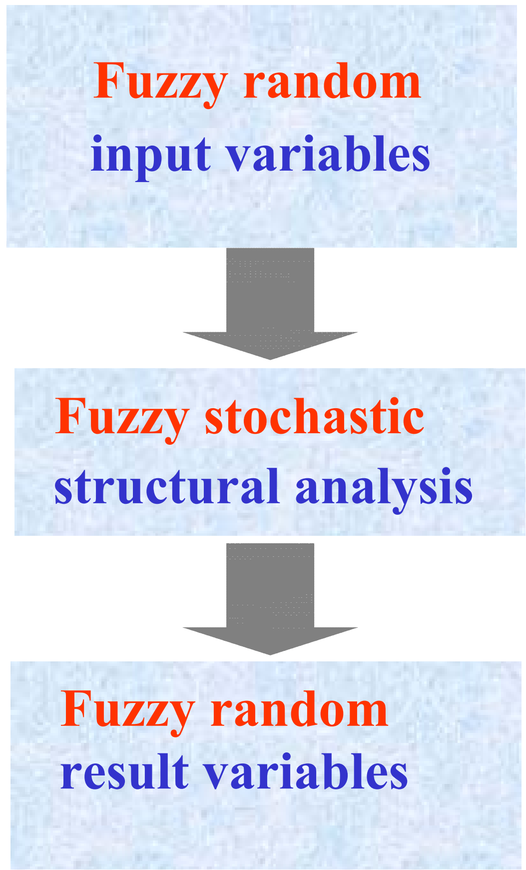
Discretization

Fuzzy Stochastic Finite Element Method (FSFEM)

$$\underline{\tilde{\mathbf{M}}}(.) \cdot \underline{\ddot{\tilde{\mathbf{v}}}} + \underline{\tilde{\mathbf{D}}}(.) \cdot \underline{\dot{\tilde{\mathbf{v}}}} + \underline{\tilde{\mathbf{K}}}(.) \cdot \underline{\tilde{\mathbf{v}}} = \underline{\tilde{\mathbf{F}}}(.)$$

$$\underline{\tilde{\mathbf{K}}}(.) \cdot \underline{\tilde{\mathbf{v}}} = \underline{\tilde{\mathbf{F}}}(.)$$

OUTLET:



1 Fuzzy random fields

2 Discretization of fuzzy random fields

3 Numerical techniques of fuzzy stochastic sampling

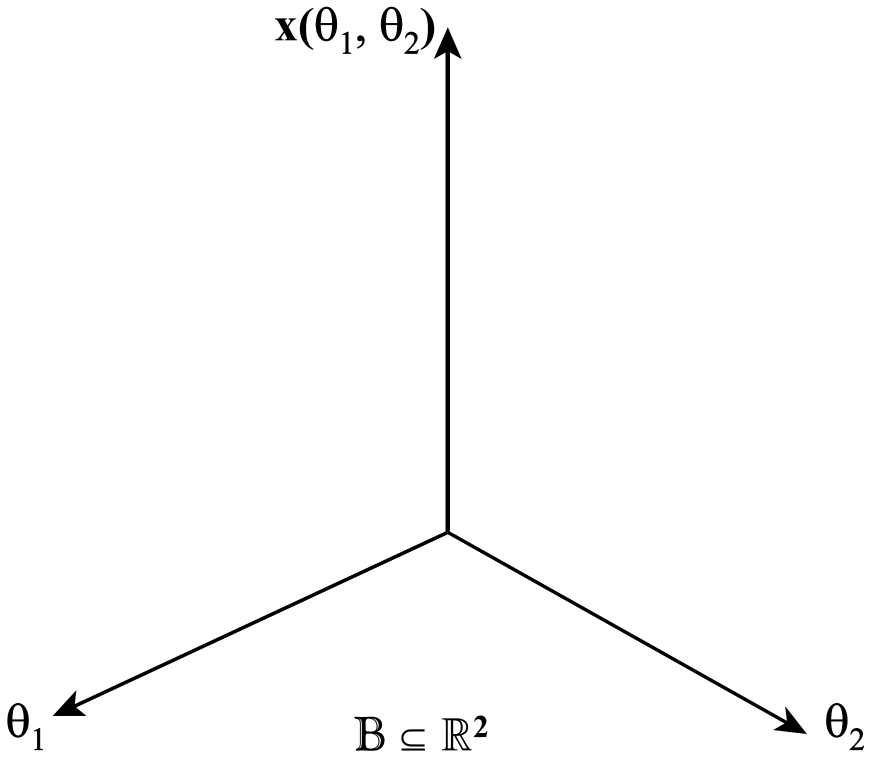
4 Result evaluation

FSFEM: Fuzzy random fields

fuzzy random field

$$\tilde{\mathbf{X}}(\underline{\theta}) = \left\{ \tilde{\mathbf{X}}_{\theta} = \tilde{\mathbf{X}}(\underline{\theta}) \;\middle|\; \underline{\theta} \in \mathbb{B} \subseteq \mathbb{R}^{\mathbf{n}} \right\}$$

time invariant fuzzy random variables



FSFEM: Fuzzy random fields

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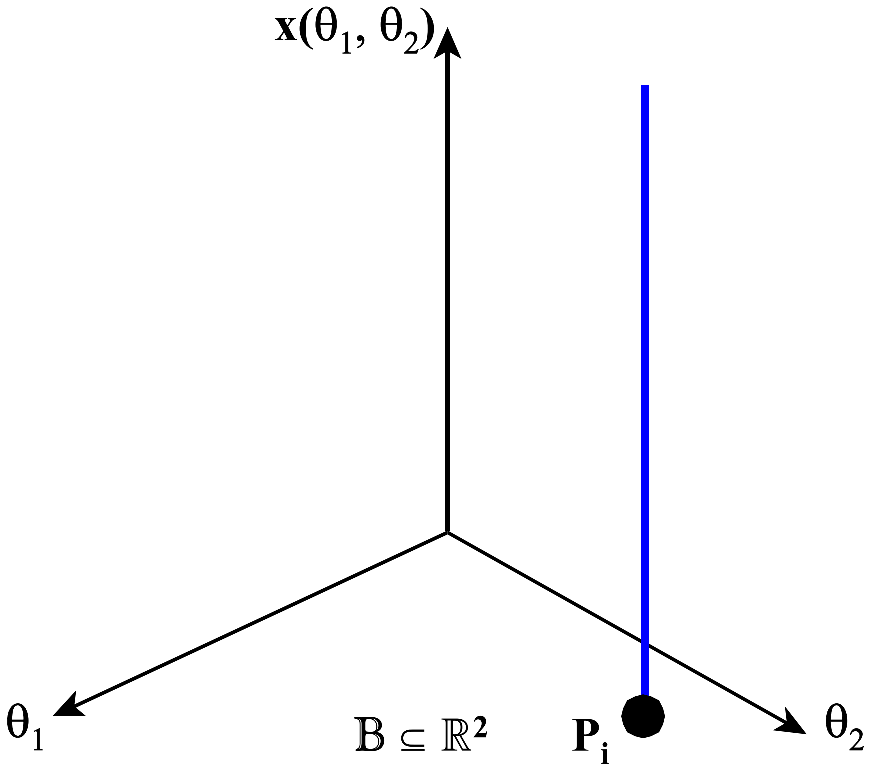
time invariant fuzzy random variables

fuzzy random variable of $P_i(\theta_1, \theta_2)$

$$\tilde{X}(\theta_1, \theta_2): \quad \Omega \tilde{\to} F(\mathbb{X}) = \left\{ \tilde{x} \;\middle|\; \tilde{x} \in \mathbb{R}^1 \right\}$$



FSFEM: Fuzzy random fields

fuzzy random field

$$\tilde{X}(\underline{\theta}) = \left\{ \tilde{X}_{\theta} = \tilde{X}(\underline{\theta}) \mid \underline{\theta} \in \mathbb{B} \subseteq \mathbb{R}^n \right\}$$

time invariant fuzzy random variables

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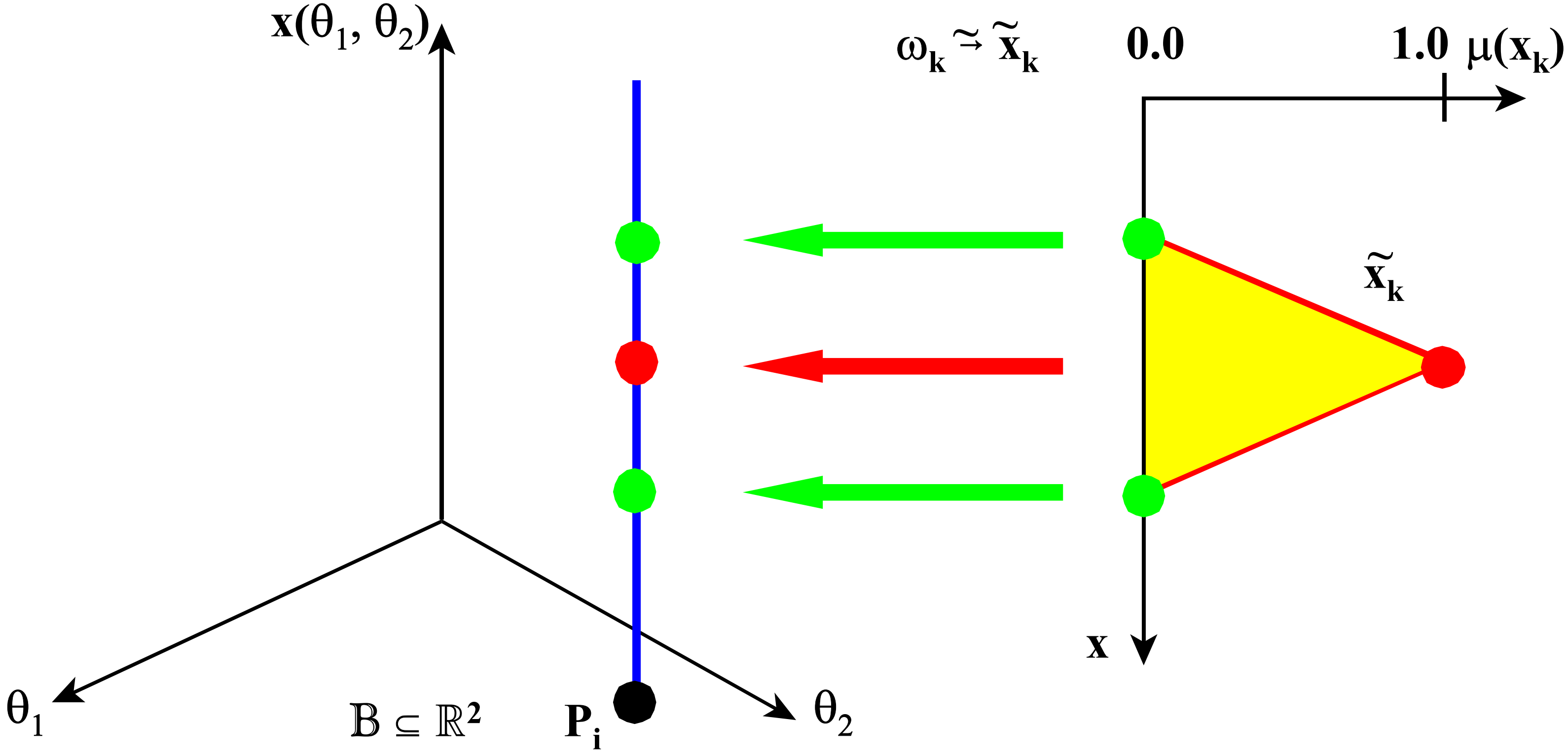
$$\tilde{X}(\theta_1, \theta_2): \quad \Omega \to F(\mathbb{X}) = \left\{ \tilde{x} \mid \tilde{x} \in \mathbb{R}^1 \right\}$$

realization of $\tilde{X}(\theta_1, \theta_2)$



FSFEM: Fuzzy random fields

fuzzy random field

$\tilde{X}(\underline{\theta}) = \{ \tilde{X}_\theta = \tilde{X}(\underline{\theta}) \mid \underline{\theta} \in \mathbb{B} \subseteq \mathbb{R}^n \}$

time invariant fuzzy random variables

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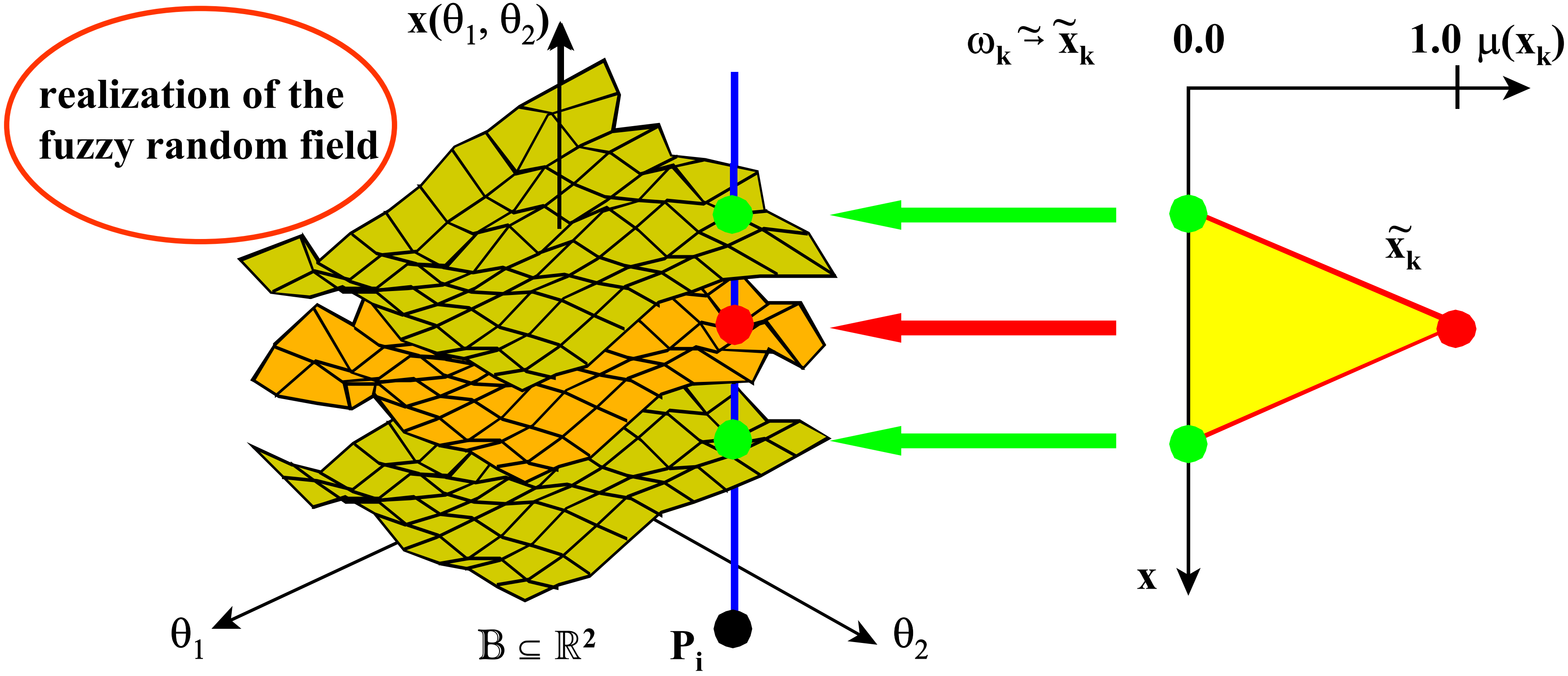
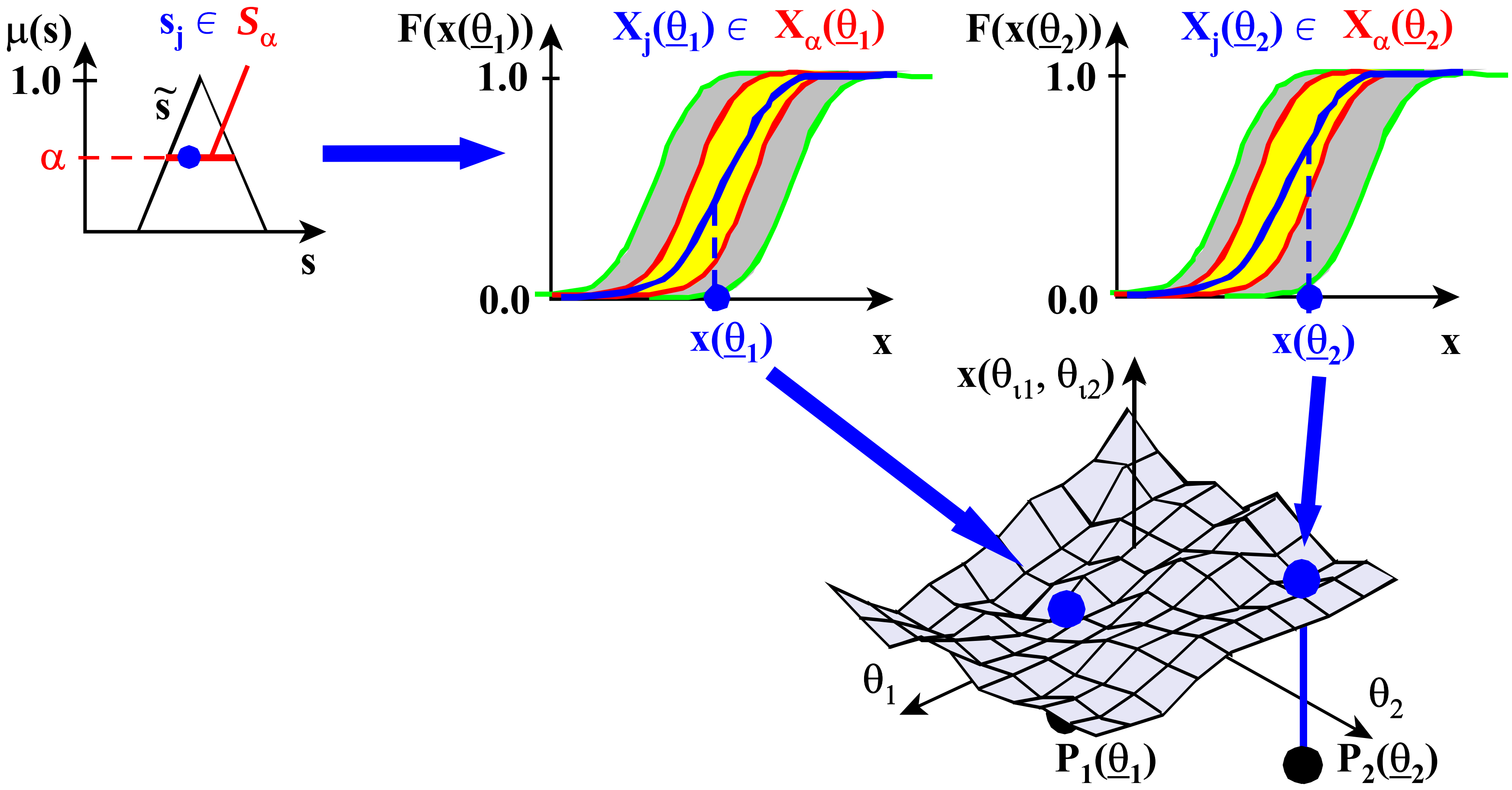
$\tilde{X}(\theta_1, \theta_2)$: $\Omega \to F(\mathbb{X}) = \{ \tilde{x} \mid \tilde{x} \in \mathbb{R}^1 \}$

realization of $\tilde{X}(\theta_1, \theta_2)$



FSFEM: Fuzzy random fields

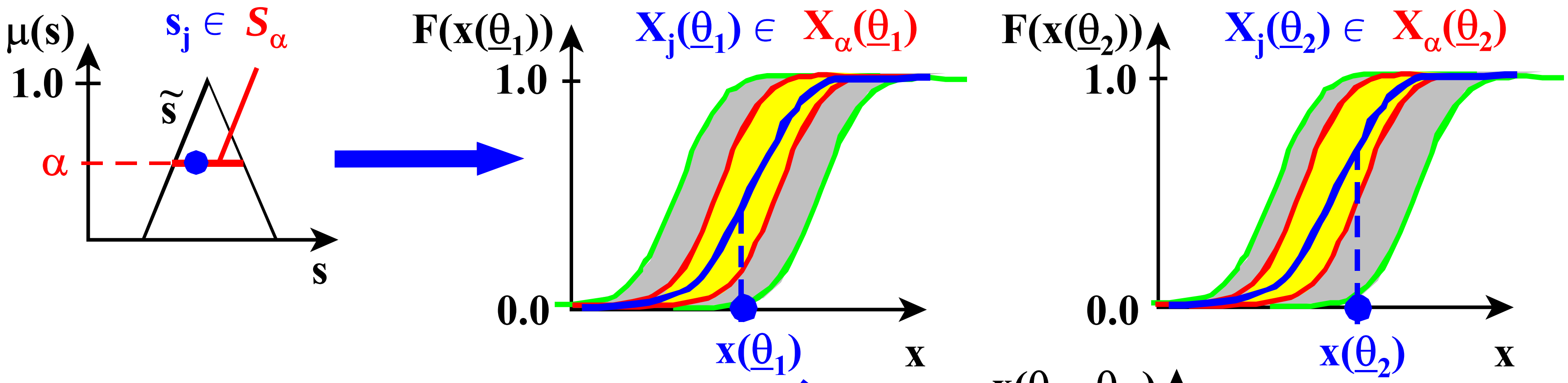
representation with fuzzy bunch parameters $\underline{\tilde{s}}$: $\tilde{X}(\underline{\theta}) = X(\underline{\tilde{s}}, \underline{\theta})$



realization of one original function of the fuzzy random field

FSFEM: Fuzzy random fields

representation with fuzzy bunch parameters $\tilde{\underline{s}}$: $\tilde{X}(\underline{\theta}) = X(\tilde{\underline{s}}, \underline{\theta})$

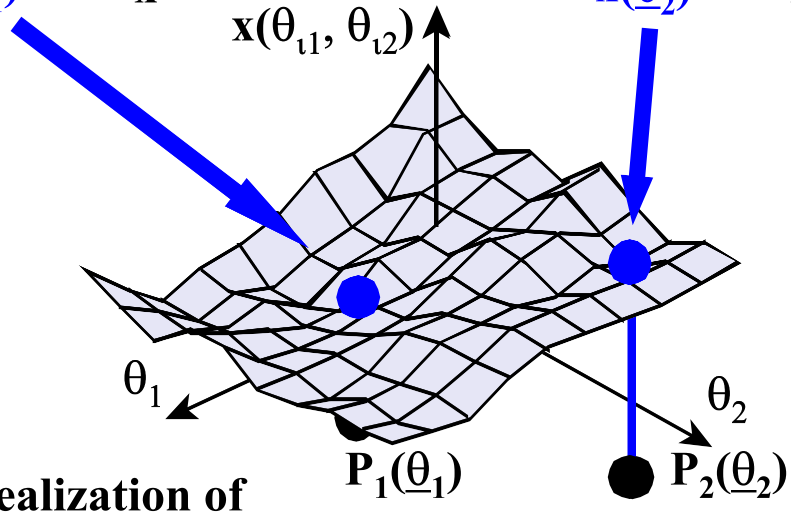


numerical evaluation with α-discretization

$$X(\tilde{\underline{s}}, \underline{\theta}) = \left\{ \left(X_\alpha(\underline{\theta}); \mu(X_\alpha(\underline{\theta})) \right) \right\}$$

$$X_\alpha(\underline{\theta}) = \left\{ \underline{X}(\underline{s}, \underline{\theta})\ \forall\ s \mid s \in S_\alpha; \alpha \in (0,1] \right\}$$

$$\mu(X_\alpha(\underline{\theta})) = \alpha\ \forall\ \alpha \in (0;1]$$



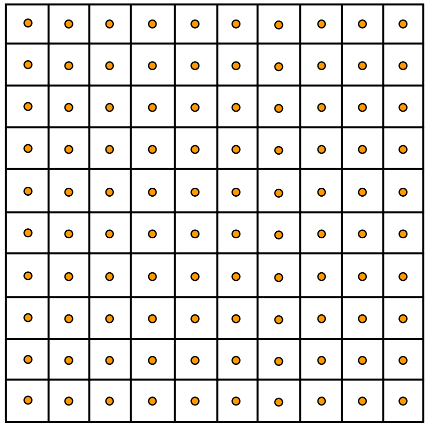
realization of one original of the fuzzy random field

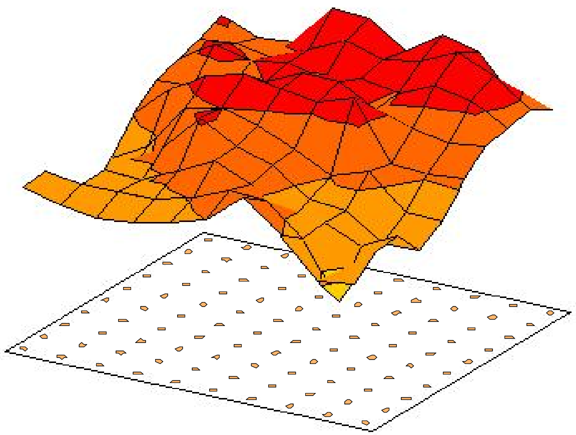
FSFEM: Fuzzy random fields

- Representation of continuous fuzzy random fields by a finite number of discrete fuzzy random variables
- Discretization

point discretization

midpoint method





nodal point method

method of local averaging

methods of weighted integrals

series extension

Karhunen-Loeve extension

FSFEM: Fuzzy random fields

Fuzzy covariance function for $\tilde{X}(\theta_1)$ and $\tilde{X}(\theta_2)$

$$\tilde{K}(\underline{\theta}_1, \underline{\theta}_2) = COV\left[\tilde{X}(\underline{\theta}_1), \tilde{X}(\underline{\theta}_2)\right]$$

Special case: stationary isotropic fuzzy random field

$$\tilde{K}(\underline{\theta}_1, \underline{\theta}_2) = \tilde{\sigma}_X^2 \cdot \tilde{k}_X(L_{12})$$

FSFEM: Fuzzy random fields

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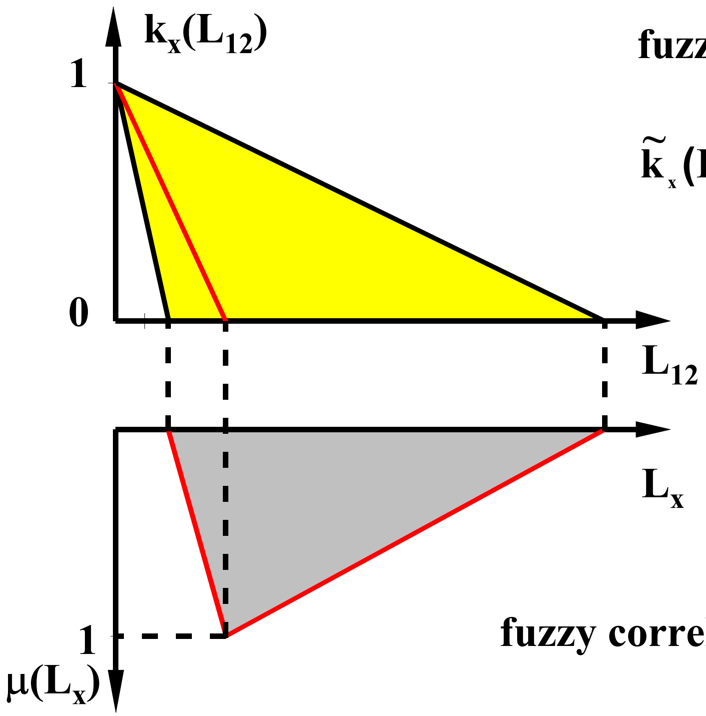

fuzzy correlation function $\tilde{k}_x(L_{12})$

$$\tilde{k}_x(L_{12}) = \begin{cases} 1 - \dfrac{L_{12}}{\tilde{L}_x} & \text{if } L_{12} \in [0, L_x] \\ 0 & \text{otherwise} \end{cases}$$

fuzzy correlation lenght $\tilde{L}_x$

FSFEM: Fuzzy random fields

Fuzzy covariance function for $\tilde{X}(\theta_1)$ and $\tilde{X}(\theta_2)$

$$\tilde{K}(\underline{\theta}_1, \underline{\theta}_2) = COV\left[\tilde{X}(\underline{\theta}_1), \tilde{X}(\underline{\theta}_2)\right]$$

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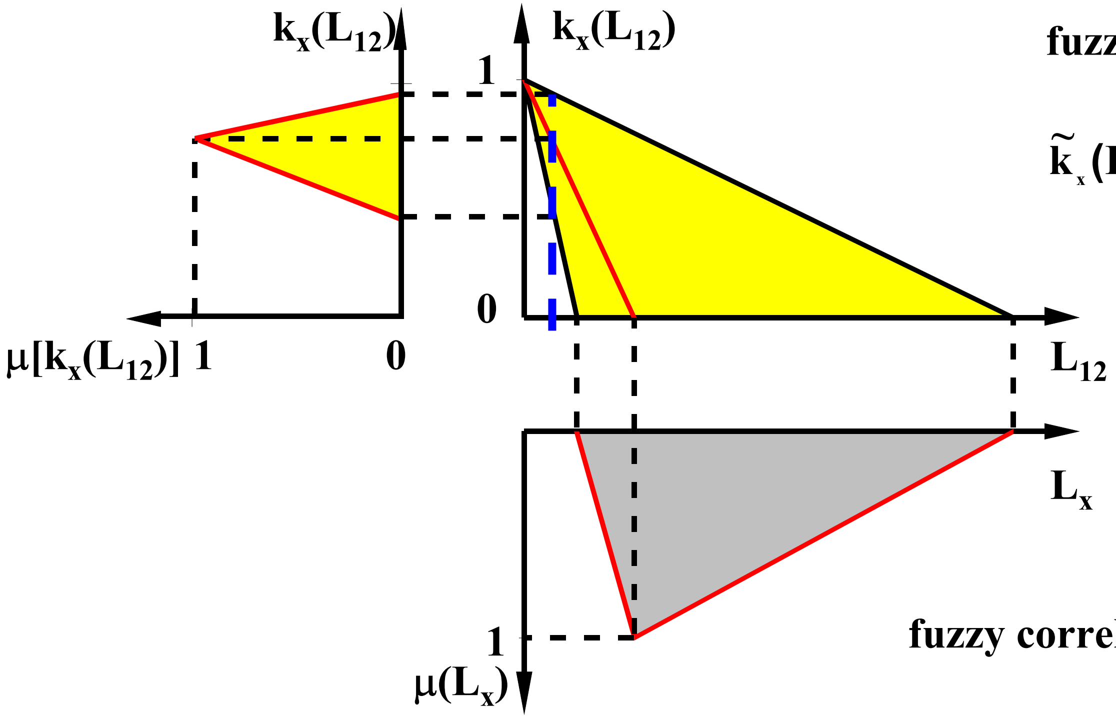

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fuzzy correlation lenght $\tilde{L}_x$

FSFEM:

OUTLET:

Fuzzy random input variables

↓

Fuzzy stochastic structural analysis

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Fuzzy random result variables

1 Fuzzy random fields

2 Representation of fuzzy random fields

3 Numerical techniques of FSFEM

4 Result evaluation

FSFEM: Numerical techniques

Construction of the bunch parameter space

Fuzzy random function

$$\tilde{F}_{\theta_i}(\underline{x}) = \tilde{F}(\underline{x}, \underline{\theta}_i) = F(\tilde{\underline{s}}_i, \underline{x}, \underline{\theta}_i) \rightarrow \begin{bmatrix} \tilde{\underline{s}}_1 \\ \vdots \\ \tilde{\underline{s}}_{p_1} \end{bmatrix} \rightarrow$$

$$i = 1, \ldots, p_1$$

$$\tilde{\underline{s}} = \begin{bmatrix} \tilde{\underline{s}}_1 \\ \vdots \\ \tilde{\underline{s}}_{n_1} \\ \\ \\ \end{bmatrix}$$

FSFEM: Numerical techniques

Construction of the bunch parameter space

Fuzzy random functions

$$\tilde{F}_{\theta_i}(\underline{x}) = \tilde{F}(\underline{x}, \underline{\theta}_i) = F(\underline{\tilde{s}}_i, \underline{x}, \underline{\theta}_i)$$

$$i = 1, \ldots, p_1$$

Fuzzy functions

$$\underline{\tilde{x}}(\underline{\theta}_i) = \underline{x}(\underline{\tilde{s}}_i, \underline{\theta}_i)$$

$$i = 1, \ldots, p_2$$

$$\begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_1} \end{bmatrix} \quad \begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_2} \end{bmatrix} \Rightarrow \underline{\tilde{s}} = \begin{bmatrix} \tilde{s}_1 \\ \vdots \\ \tilde{s}_{n_1} \\ \\ \\ \\ \end{bmatrix}$$

FSFEM: Numerical techniques

Construction of the bunch parameter space

Fuzzy random functions

$$\tilde{F}_{\theta_i}(\underline{x}) = \tilde{F}(\underline{x}, \underline{\theta}_i) = F(\underline{\tilde{s}}_i, \underline{x}, \underline{\theta}_i)$$

$$i = 1, \dots, p_1$$

$$\begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_1} \end{bmatrix}$$

Fuzzy functions

$$\underline{\tilde{x}}(\underline{\theta}_i) = \underline{x}(\underline{\tilde{s}}_i, \underline{\theta}_i)$$

$$i = 1, \dots, p_2$$

$$\begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_2} \end{bmatrix}$$

Random functions

$$F_{\theta_i}(\underline{x}) = F(\underline{x}, \underline{\theta}_i) = F(\underline{s}_i, \underline{x}, \underline{\theta}_i)$$

$$i = 1, \dots, p_3$$

$$\rightarrow \begin{bmatrix} \underline{s}_1 \\ \vdots \\ \underline{s}_{p_3} \end{bmatrix} \rightarrow$$

$$\underline{\tilde{s}} = \begin{bmatrix} \tilde{s}_1 \\ \vdots \\ \tilde{s}_{n_1} \\ \tilde{s}_{n_1+1} \\ \vdots \\ \tilde{s}_{n_1+n_2} \\ \\ \\ \end{bmatrix}$$

FSFEM: Numerical techniques

Construction of the bunch parameter space

Fuzzy random functions

$$\tilde{F}_{\theta_i}(\underline{x}) = \tilde{F}(\underline{x}, \underline{\theta}_i) = F(\underline{\tilde{s}}_i, \underline{x}, \underline{\theta}_i)$$

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Fuzzy functions

$$\underline{\tilde{x}}(\underline{\theta}_i) = \underline{x}(\underline{\tilde{s}}_i, \underline{\theta}_i)$$

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Random functions

$$F_{\theta_i}(\underline{x}) = F(\underline{x}, \underline{\theta}_i) = F(\underline{s}_i, \underline{x}, \underline{\theta}_i)$$

$$i = 1, \ldots, p_3$$

Dependencies

$$\tilde{k}_{x_i}(L_{12}) = k_x(\underline{\tilde{s}}_i, L_{12})$$

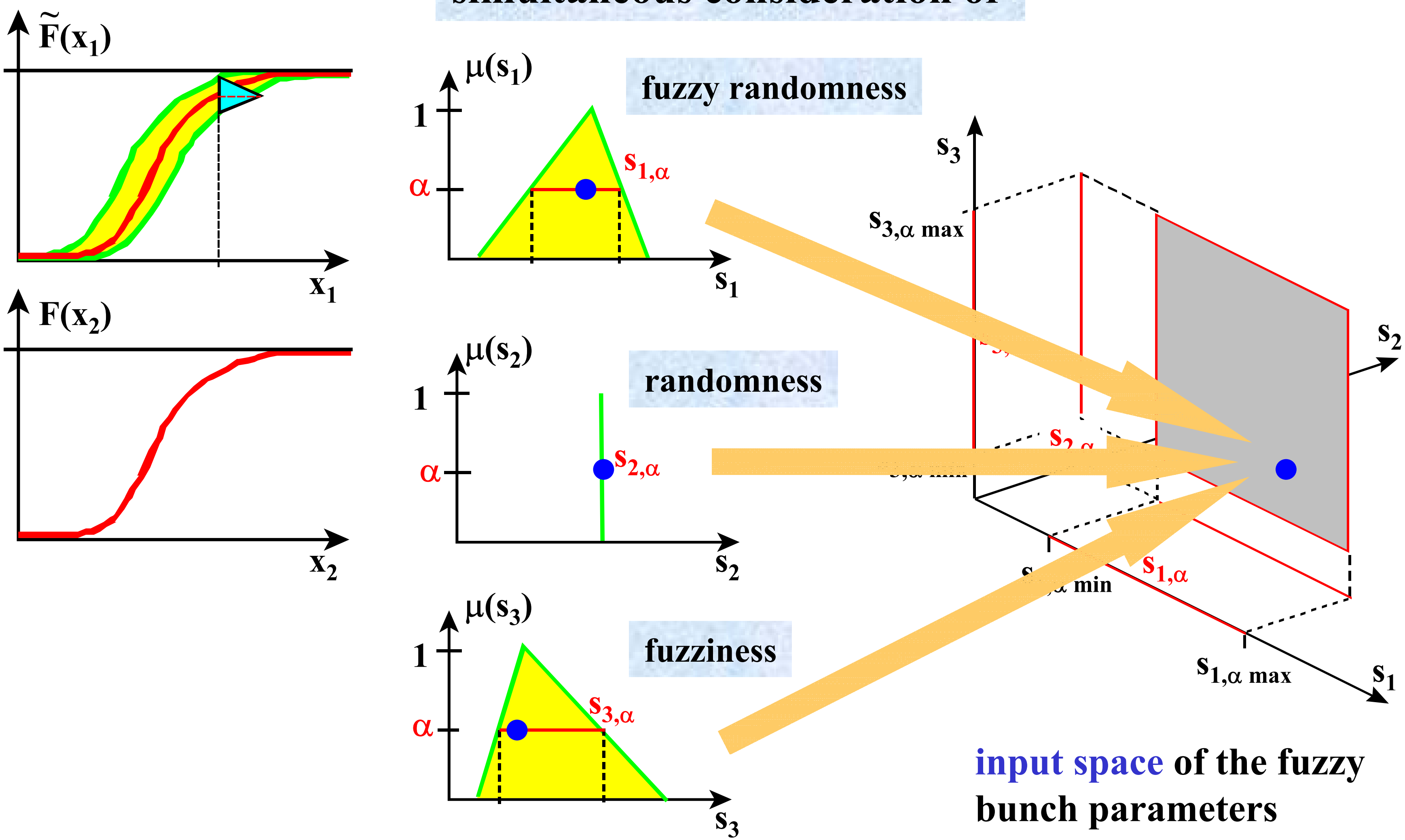
$$i = 1, \ldots, p_4$$

$$\begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_1} \end{bmatrix} \quad \begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_2} \end{bmatrix} \quad \begin{bmatrix} \underline{s}_1 \\ \vdots \\ \underline{s}_{p_3} \end{bmatrix} \quad \begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_4} \end{bmatrix}$$

$$\underline{\tilde{s}} = \begin{bmatrix} \tilde{s}_1 \\ \vdots \\ \tilde{s}_{n_1} \\ \tilde{s}_{n_1+1} \\ \vdots \\ \tilde{s}_{n_1+n_2} \\ s_{n_1+n_2+1} \\ \vdots \\ s_{n_1+n_2+n_3} \\ \\ \end{bmatrix}$$

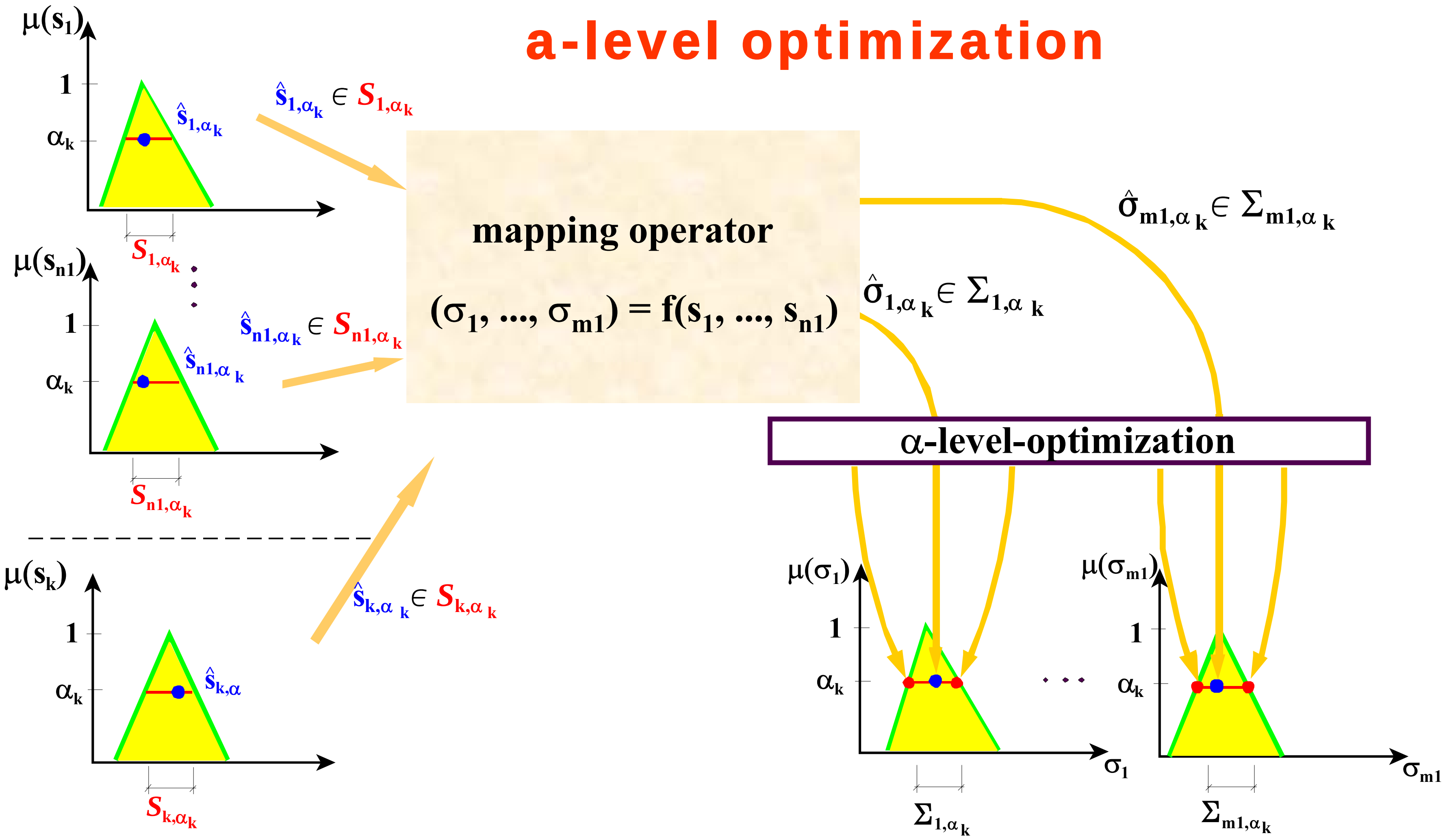
FSFEM: Numerical techniques

simultaneous consideration of

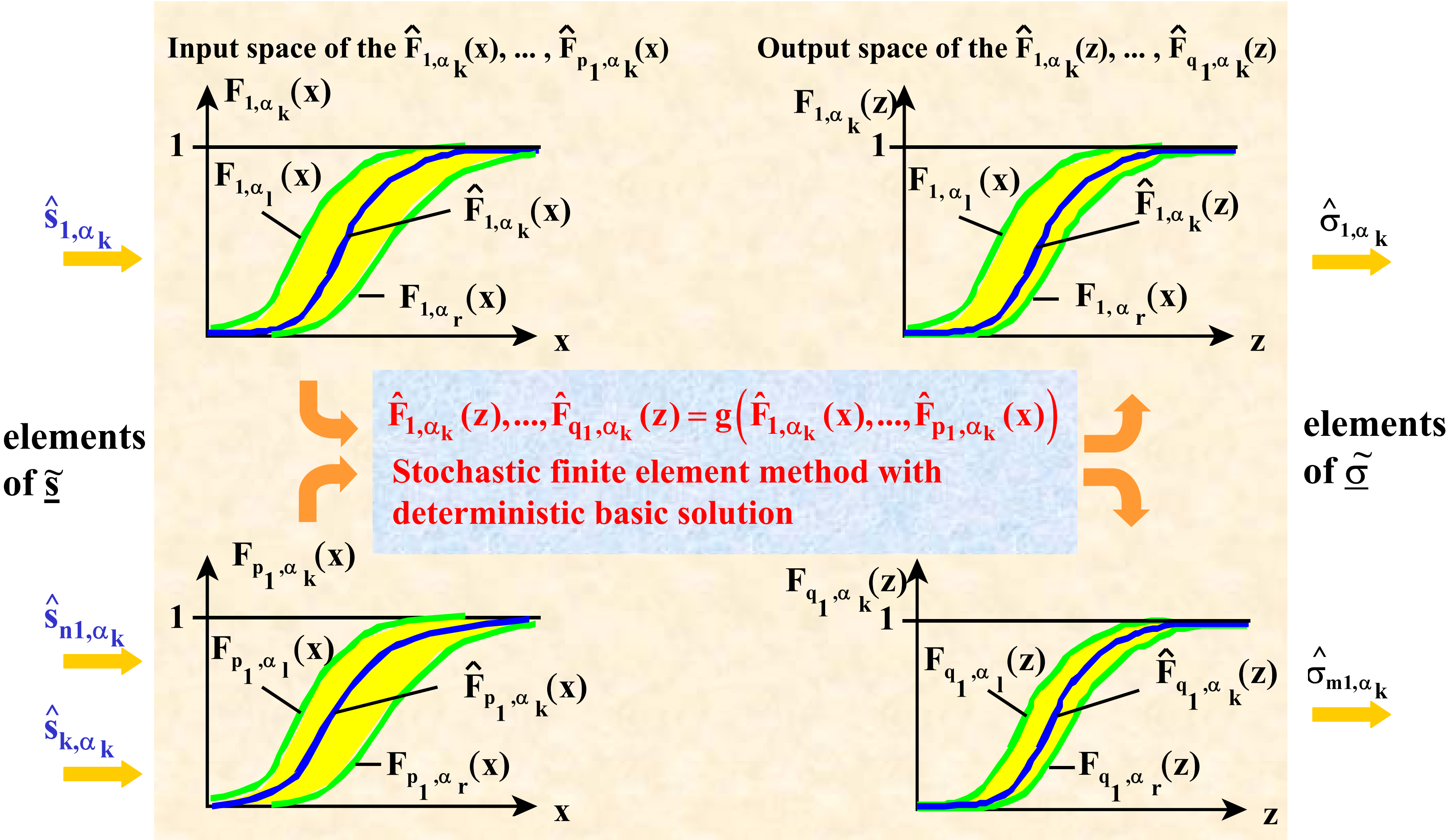


input space of the fuzzy bunch parameters

a-level optimization



Mapping model



FSFEM: Numerical techniques

$$\hat{F}_{1,\alpha_k}(z),\ldots,\hat{F}_{q1,\alpha_k}(z) = g\left(\hat{F}_{1,\alpha_k}(x),\ldots,\hat{F}_{p1,\alpha_k}(x)\right)$$

Stochastic finite element method with deterministic FEM algorithm

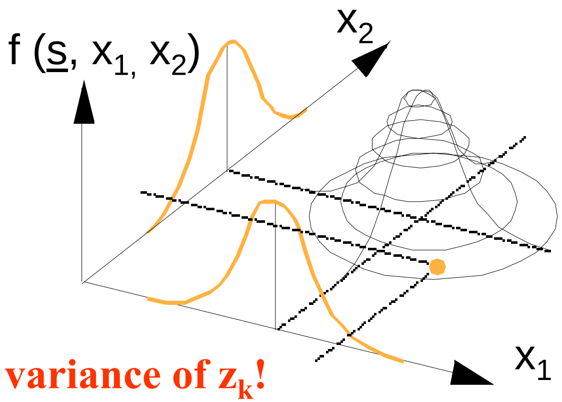
Perturbation methods

Polynomial Chaos

Monte-Carlo Simulation

generation of n crisp realizations for all input variables

computing of n result variables z_k

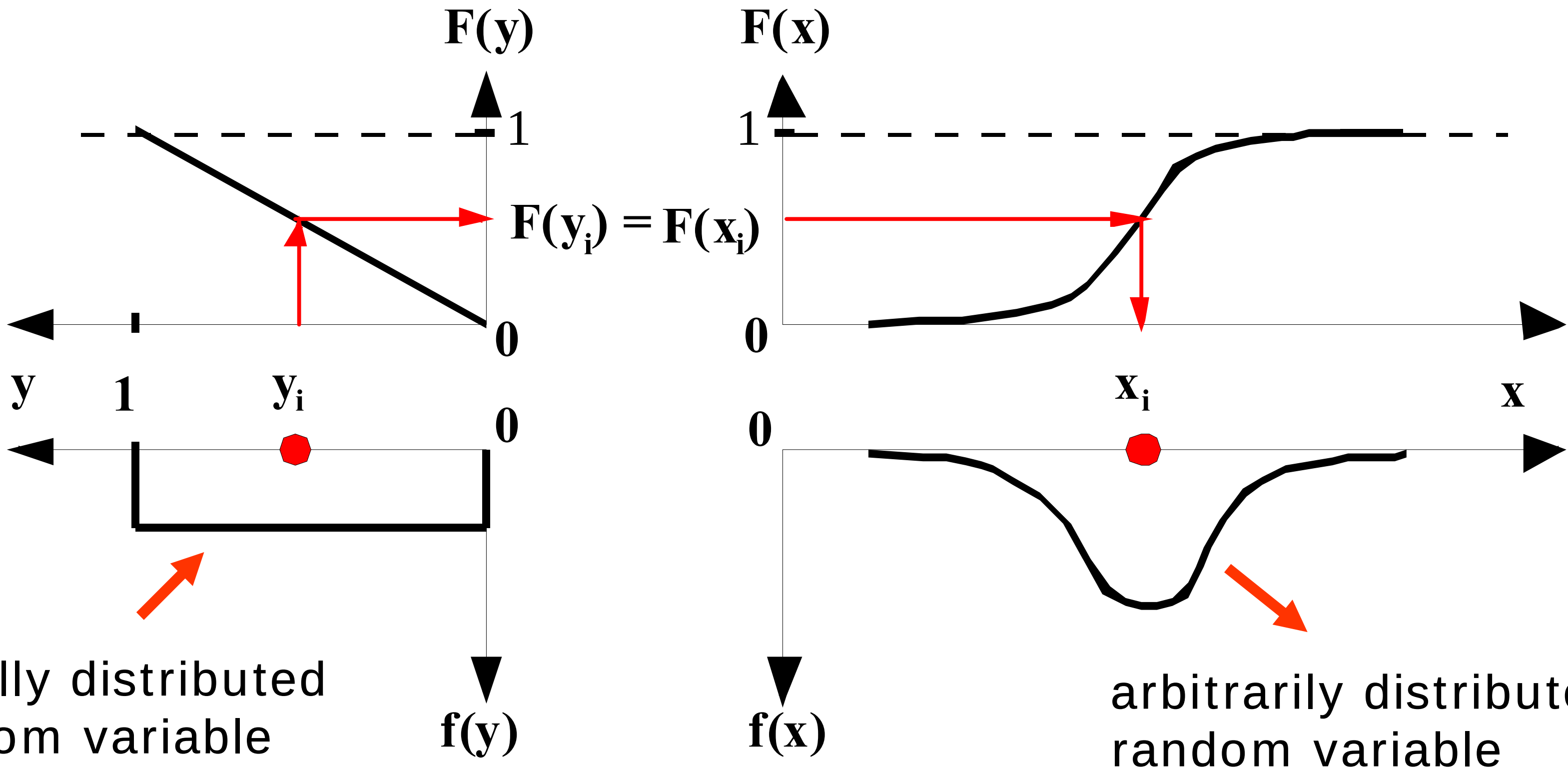


checking of the variance of z_k!

Monte Carlo Simulation (1)

Method of inverse distribution function

$$X = F_X^{-1}(Y)$$



equally distributed random variable

arbitrarily distributed random variable

FSFEM:

OUTLET:

Fuzzy random input variables

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Fuzzy stochastic structural analysis

↓

Fuzzy random result variables

1 Fuzzy random fields

2 Representation of fuzzy random fields

3 Numerical techniques of FSFEM

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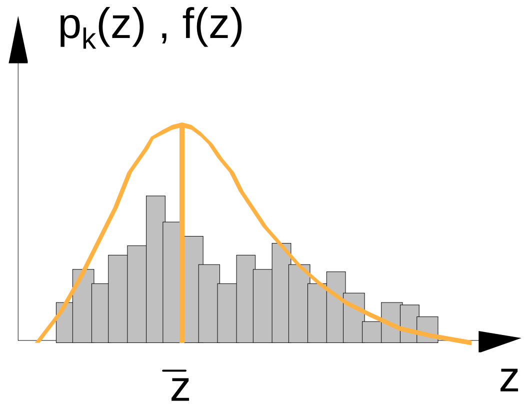
FSFEM: Result evaluation

- **random sample with n sample elements for each result values z**
- **parameter estimation of the sample parameter (mean value, variance)**
- **estimation of quantiles of the sample**
- **empirical distribution function**
- **test of different typs of probability distribution functions – determination of the parameter**

bunch parameter $\hat{\underline{\sigma}}$

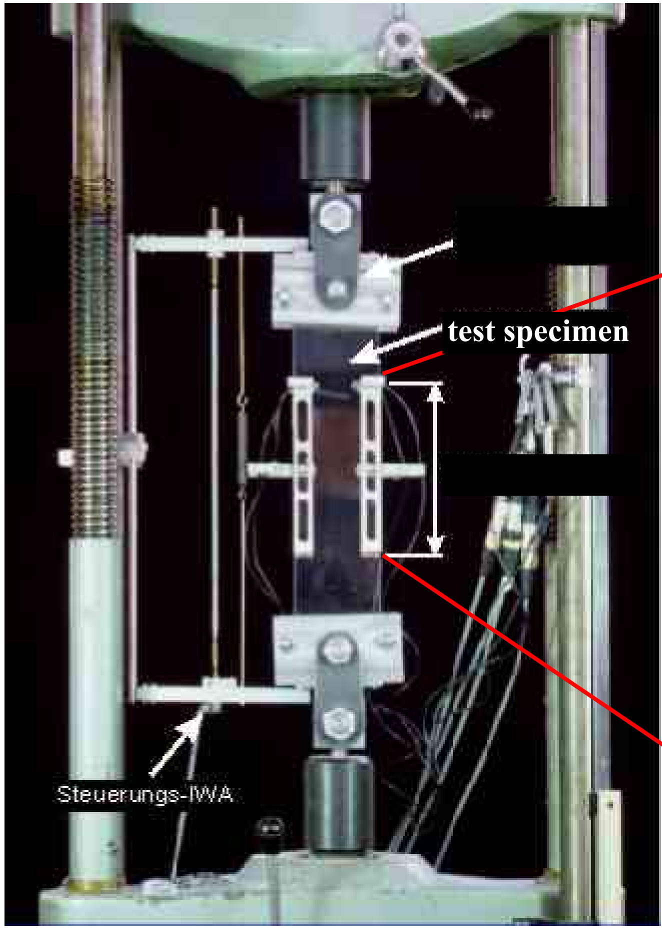
histogram

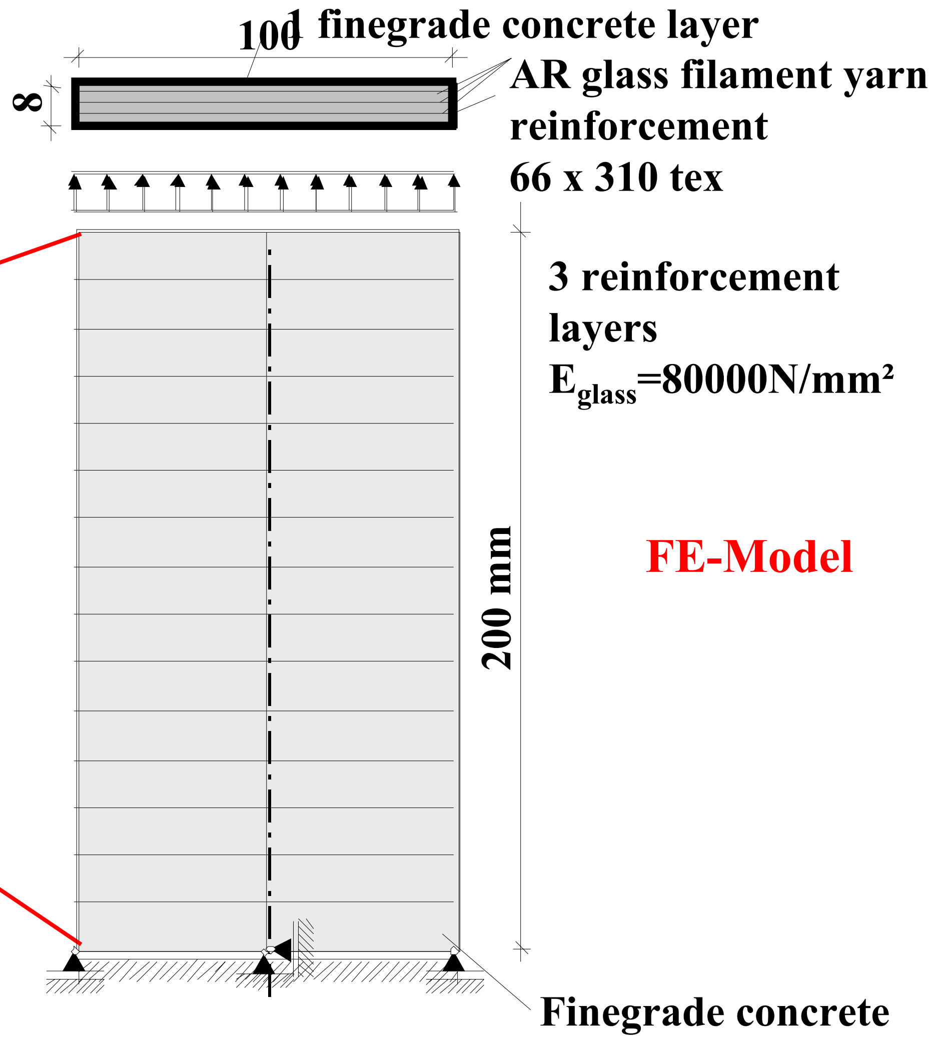


$$\bar{z} = \sum_n z_k \cdot p_k$$

$$\sigma_z = \sqrt{\sum_n (z_k - \bar{z}) \cdot p_k}$$

Example: textile reinforced test specimen (1)



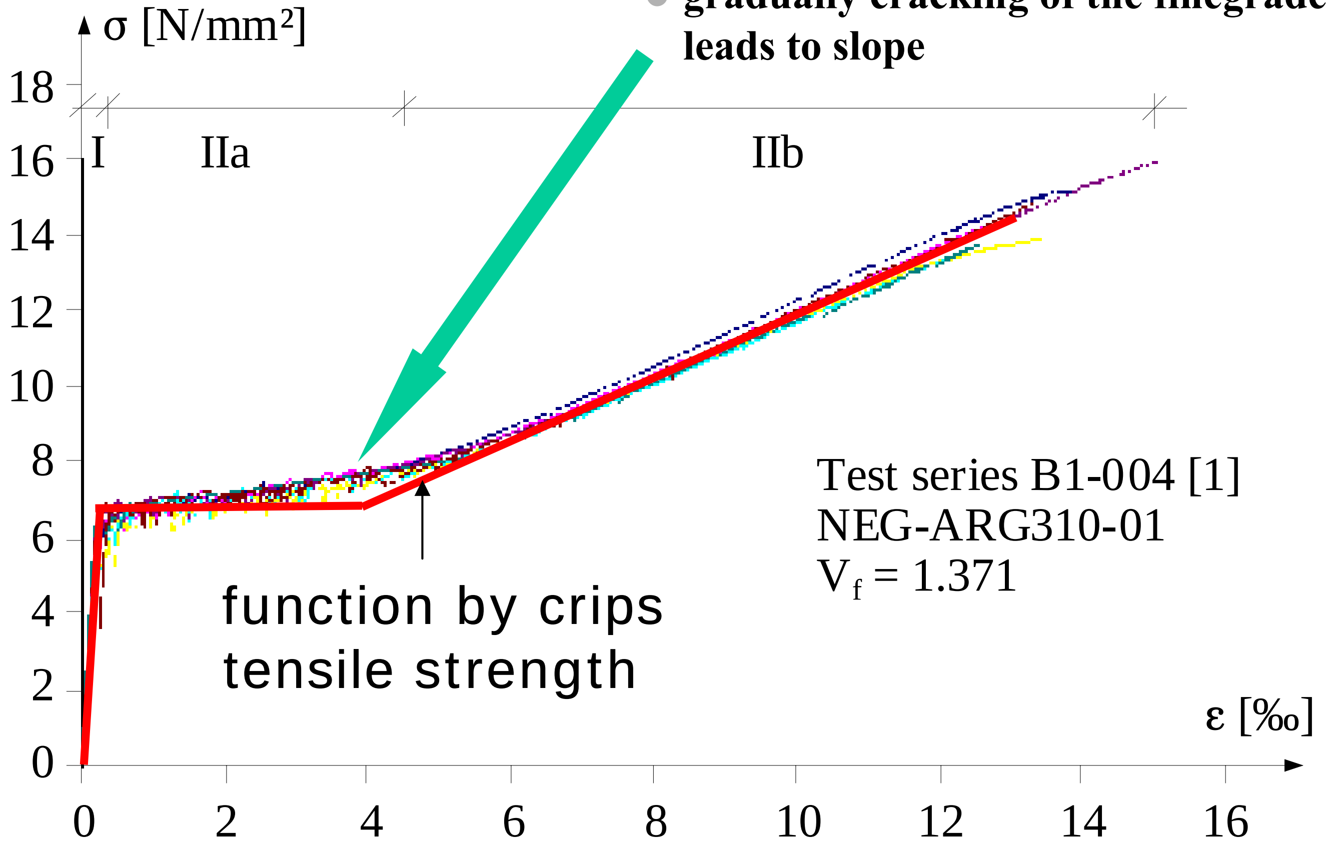


assumption of homogeneous material leads to simultaneous cracking → wrong

Example: textile reinforced test specimen (2)

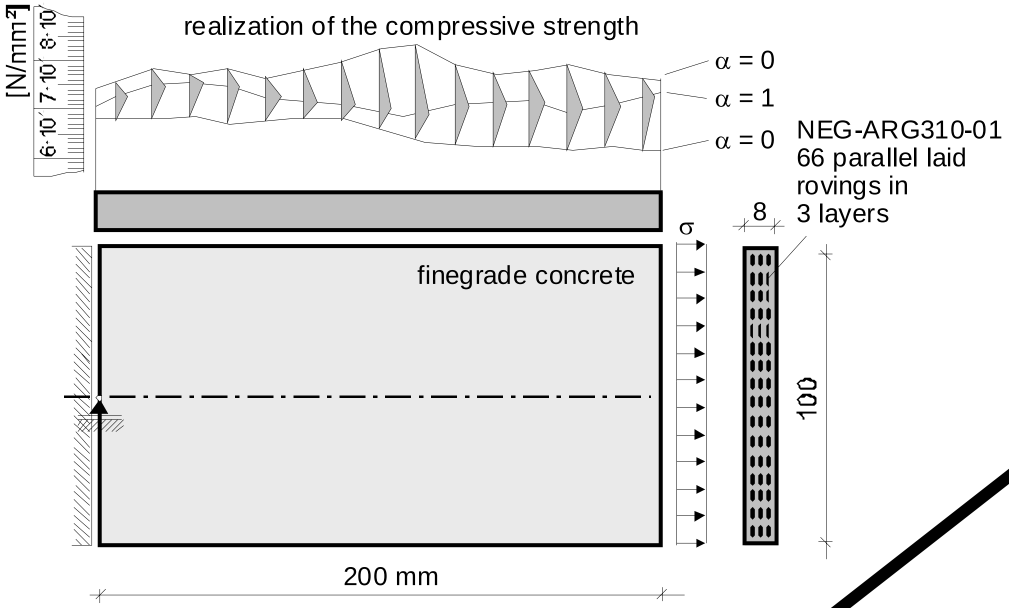
uncertainty of material parameters as consquence of:

- **different stress-strain curves for different tests**
- **gradually cracking of the finegrade concrete leads to slope**



Example: textile reinforced test specimen (3)

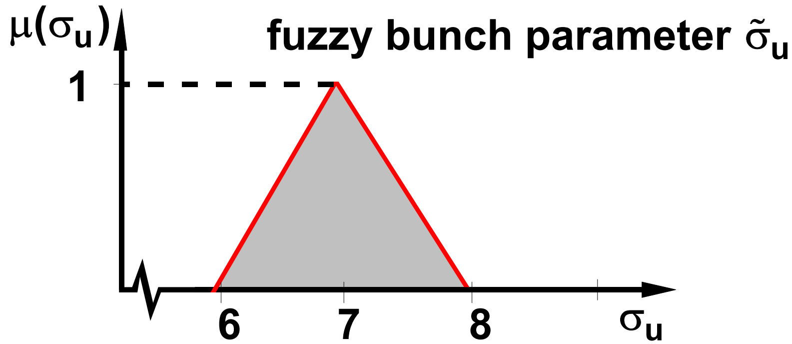
Tensile strength and compressive strength are correlated caused by the material law of concrete.



Compressive strength are modeled as <u>stationary</u> fuzzy random field.

Discretization yields 10 fuzzy random variables with equal fuzzy probality distribution function.

$$F_i\left(x, \tilde{s} = \tilde{\sigma}_u\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln x - \ln 6.268 + \tilde{\sigma}_u^2/2}{\tilde{\sigma}_u}} \exp\left(-\frac{u^2}{2}\right) du$$



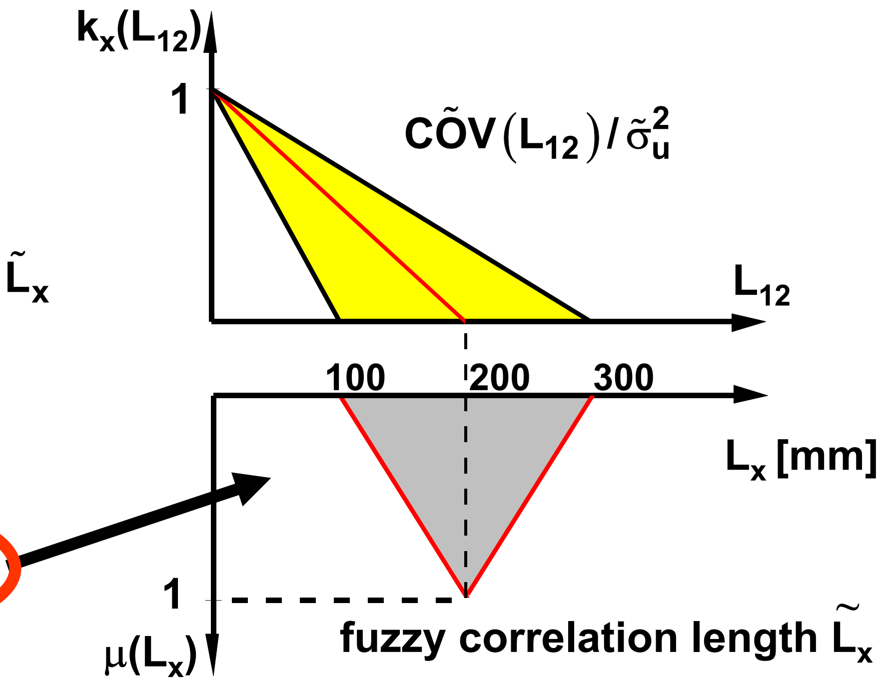
Example: textile reinforced test specimen (4)

Correlation between the 10 fuzzy random variables are described by the linear fuzzy-covariance function.

$$\mathrm{COV}(X_1, X_2) = E\left[\left(X_1 - E(X_1)\right)\cdot\left(X_2 - E(X_2)\right)\right]$$

$$\tilde{\mathrm{COV}}(X_1, X_2) = \tilde{\sigma}_u^2 \cdot \begin{cases} 1 - \dfrac{L_{12}}{\tilde{L}_x} & \text{if } L \in [0, L_x]\ \forall L_x \in \tilde{L}_x \\ 0 & \text{otherwise} \end{cases}$$

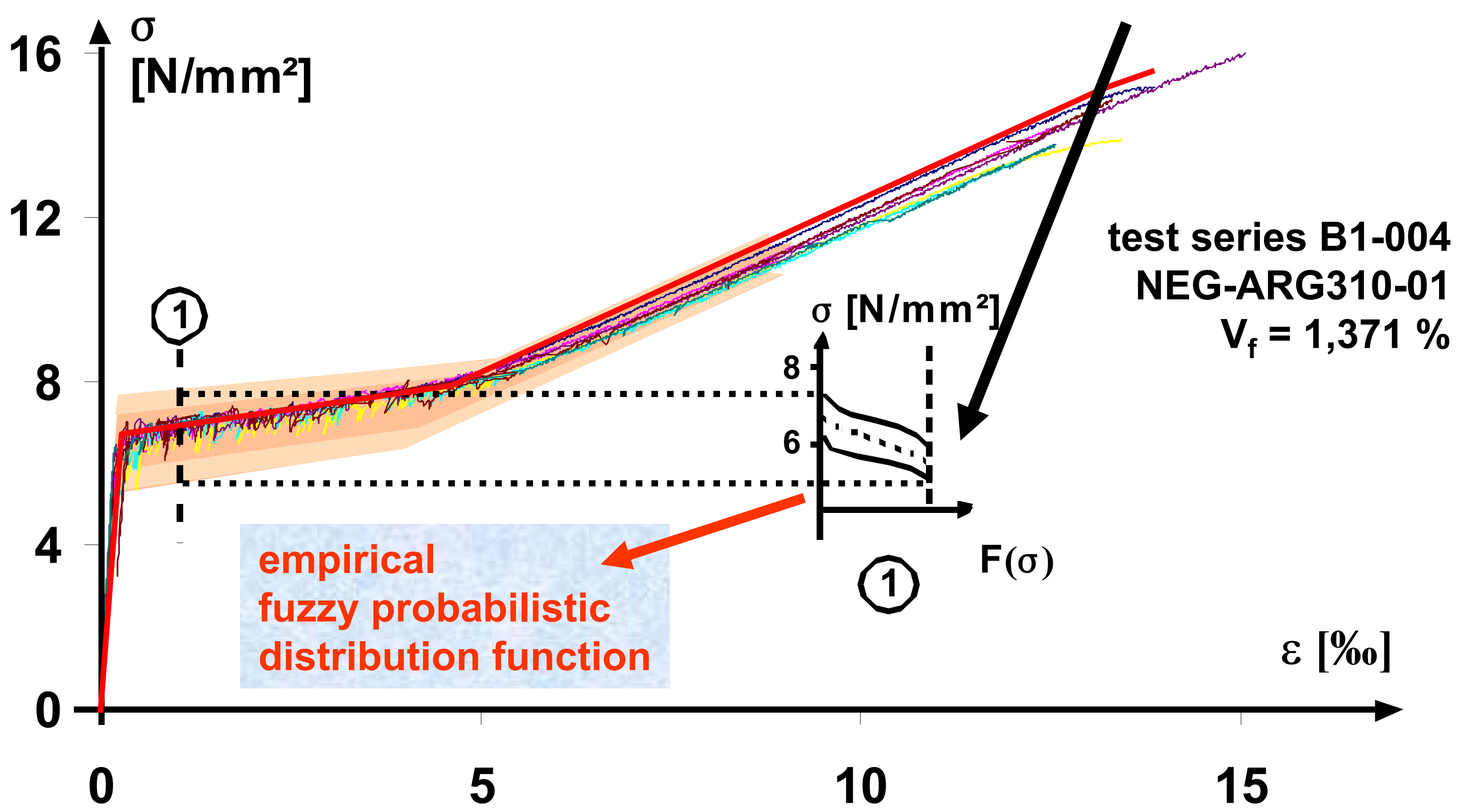
bunch parameter



Example: textile reinforced test specimen (5)

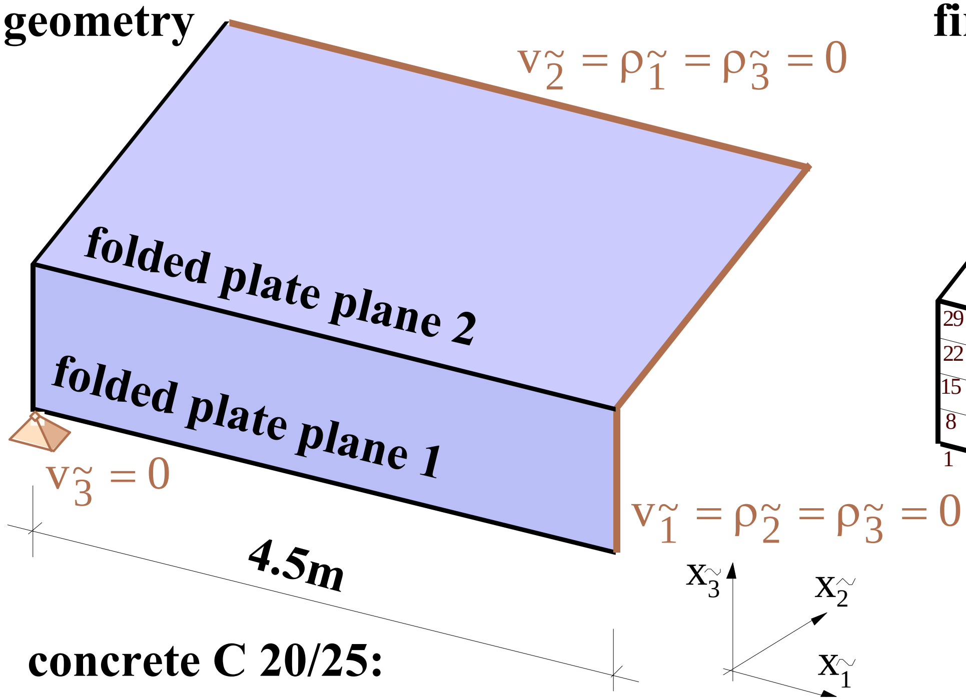
Simulation with FSFEM yields fuzzy random function for stress strain dependency.

Functional values are fuzzy random values with fuzzy probability distribution function

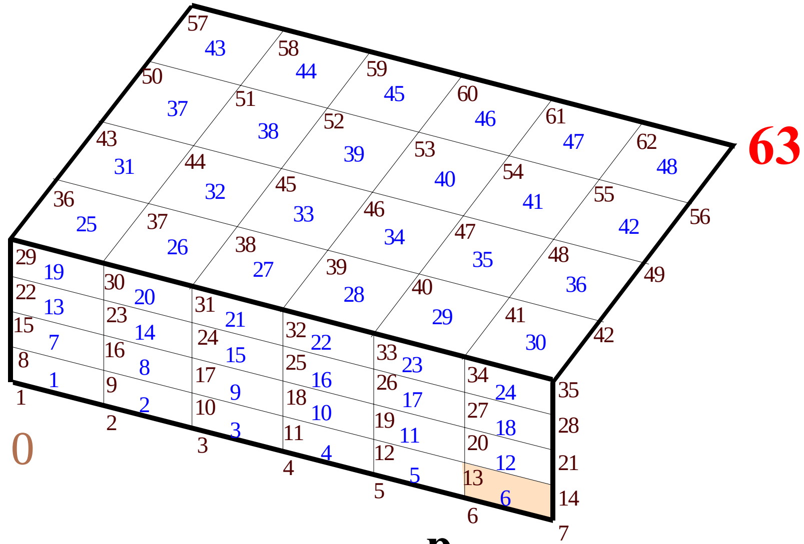


Example: reinforced folded plate structure (1)

geometry



finite element model

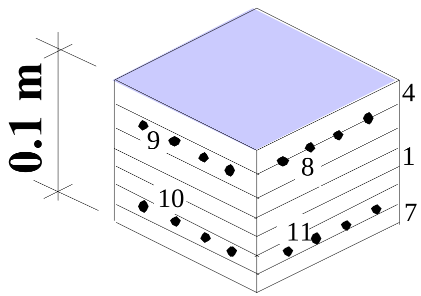


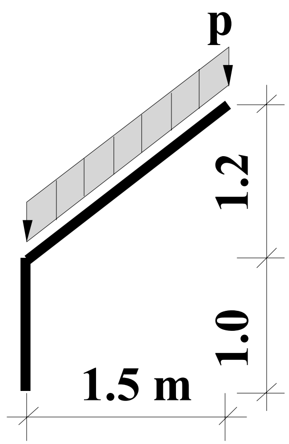
concrete C 20/25:

layer 1-7

reinforcement S 500:

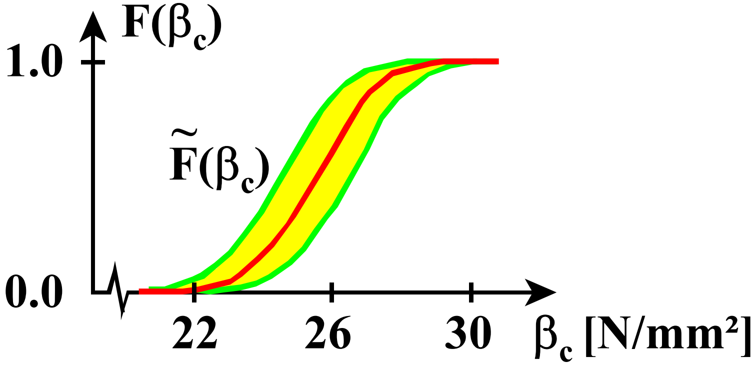
layer 8	7.85 cm²/m
layer 9	2.52 cm²/m
layer 10	2.52 cm²/m
layer 11	7.85 cm²/m



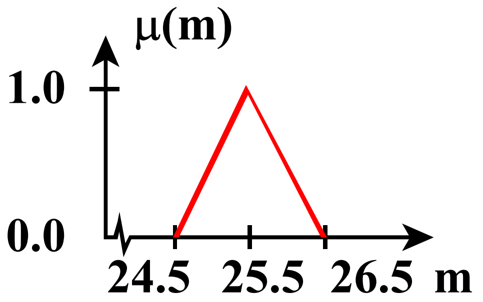


Example: reinforced folded plate structure (2)

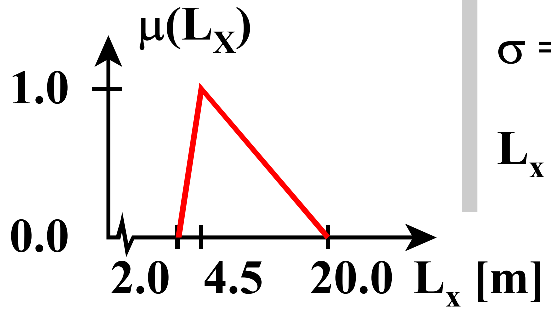
stationary isotropic fuzzy random field for concrete compressive strength



$$\tilde{F}(\beta_c) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-0.5 \left(\frac{t - \tilde{m}}{\sigma^2} \right)^2 \right) dt, \quad \sigma = 1.5$$

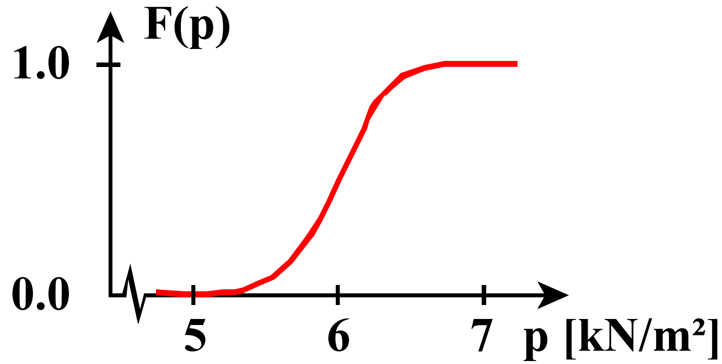


fuzzy expected value $\tilde{m}$



fuzzy correlation length $\tilde{L}_x$

random field for superficial load



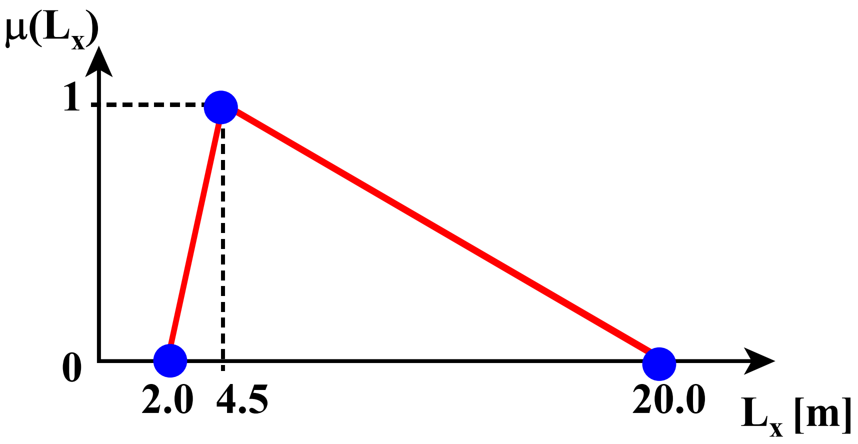
$$F(p) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-0.5 \left(\frac{t - m}{\sigma^2} \right)^2 \right) dt$$

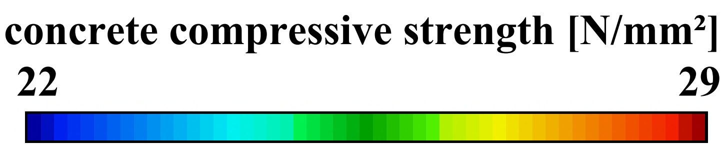
$\sigma = 0.3$, $m = 6.0$

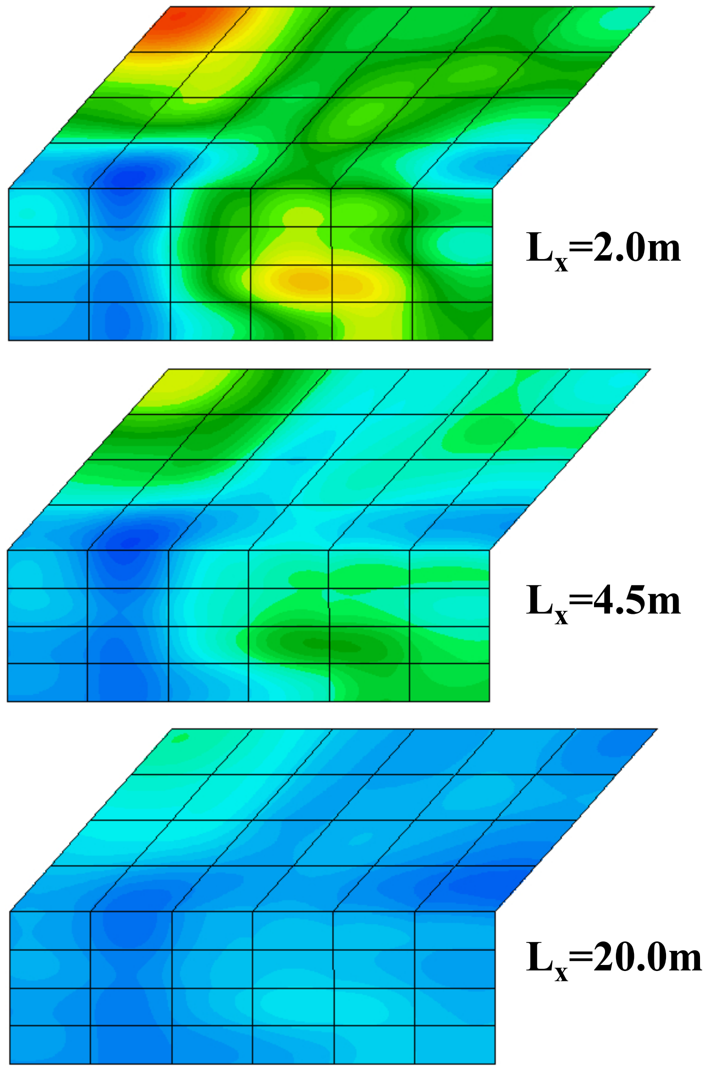
$L_x = \infty$, perfect correlated random field

Example: reinforced folded plate structure (3)

realizations of the fuzzy random field
in dependency of the
fuzzy correlation length $\tilde{L}_x$







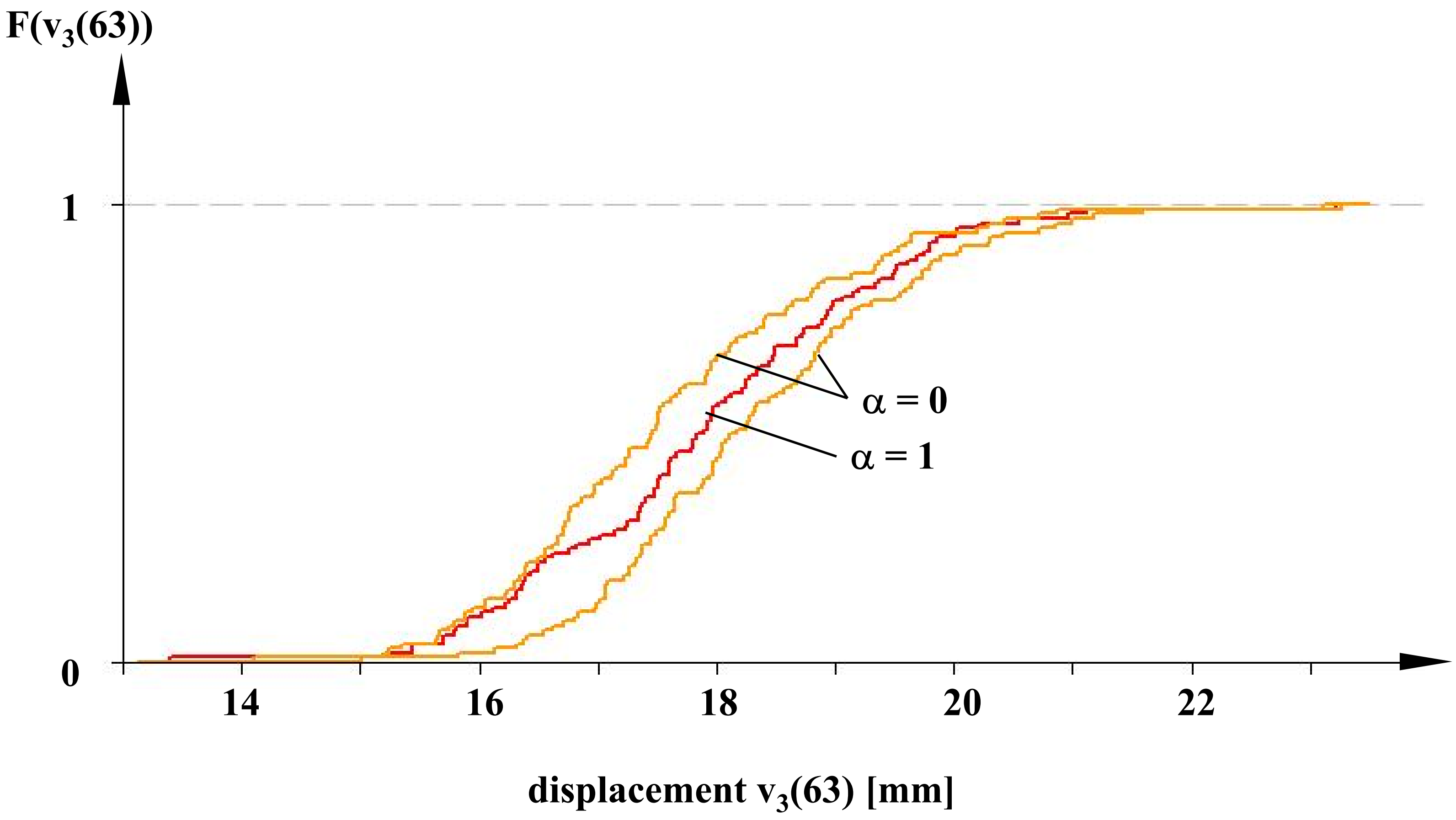
Example: reinforced folded plate structure (4)

features of the deterministic finite element algorithm

- **physical nonlinear model on layer-to-layer basis**
- **consideration of the governing nonlinearities of reinforced concrete**
  - cracking
  - tension stiffening
  - nonlinear material laws for concrete and steel
- **incremental iterative solution strategy**

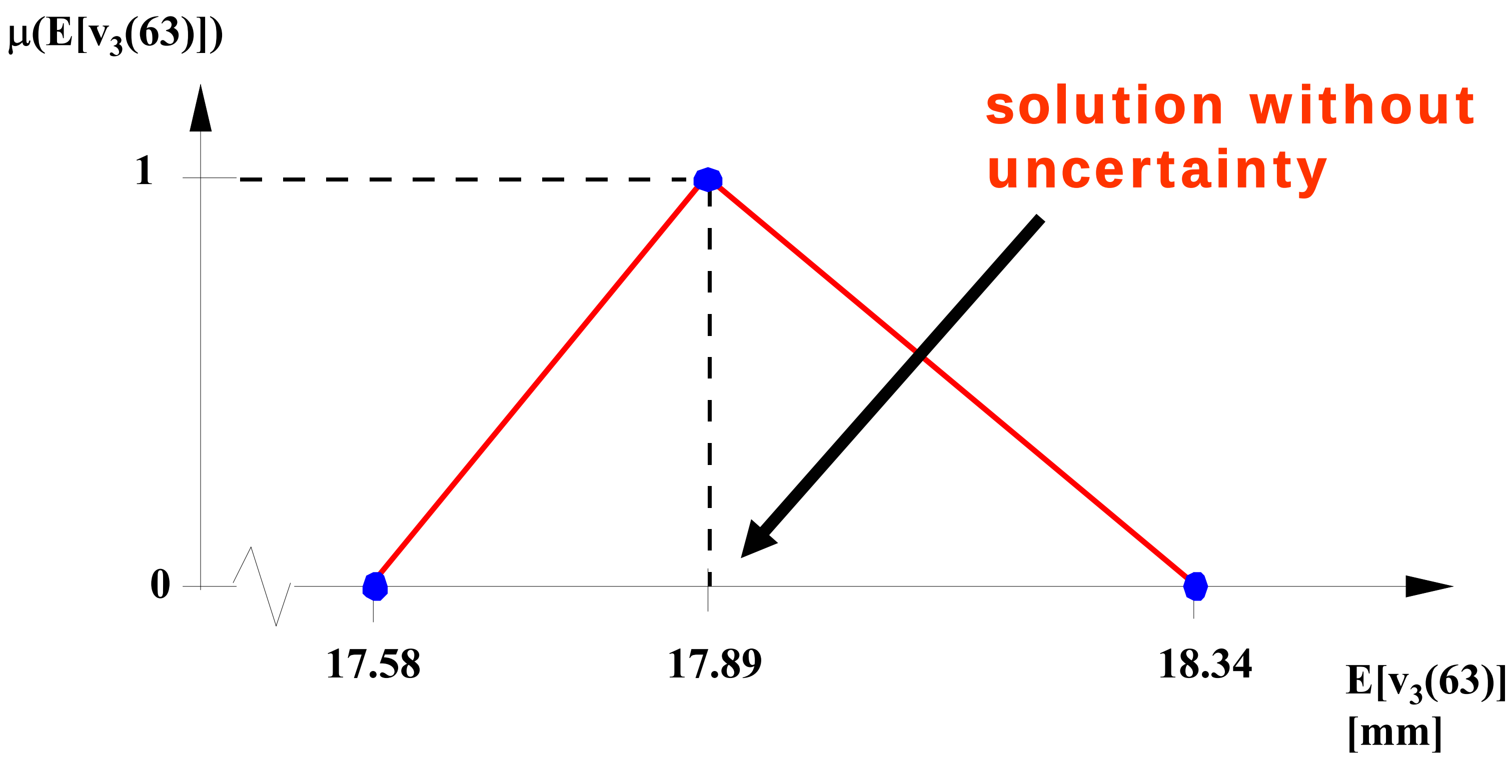
Example: reinforced folded plate structure (5)

empirical fuzzy probability distribution function for displacement $v_{63}(63)$



Example: reinforced folded plate structure (6)

fuzzy expected value of the displacement $v_3(63)$



Thank you !