TECHNISCHE UNIVERSITÄT DRESDEN

Institute for Statics und Dynamics of Structures

Mathematical Basics

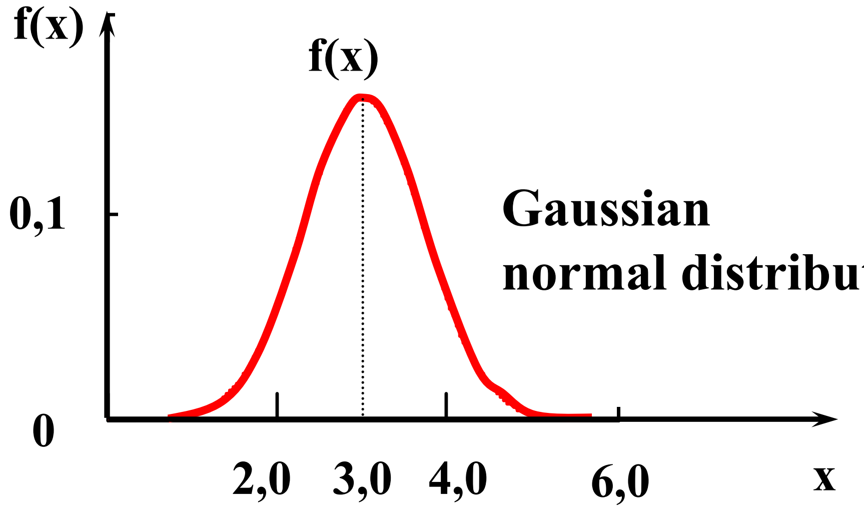
of Fuzzy Randomness

Bernd Möller

Sample Simulation (1)

F(x)

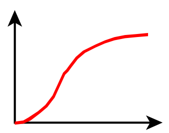
univers $\sim N(\mu = 3, \sigma = 1)$



Generating of samples with 100 sample elements

Sample Simulation (1)

- $F(x)$
- $X_1, \ldots, X_n$
- $F(x; \theta)$
- $\hat{\theta}_j$
- $E[X]$, $Var[X]$, q_α



universe $\sim N(\mu = 3, \sigma = 1)$

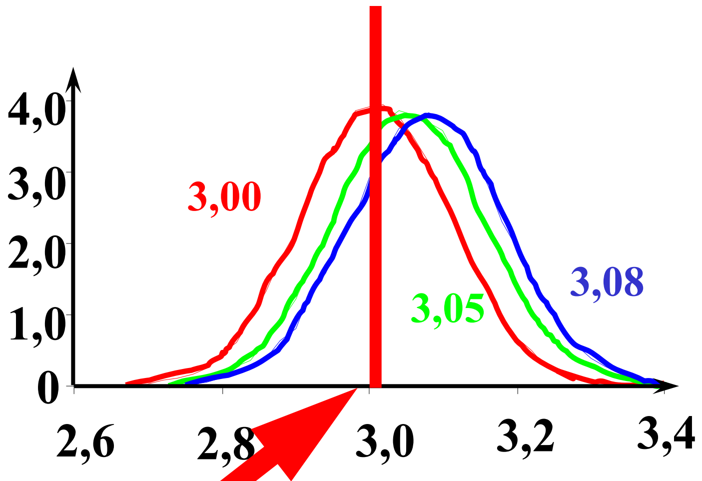
sampling

40,000 samples of size n = 100

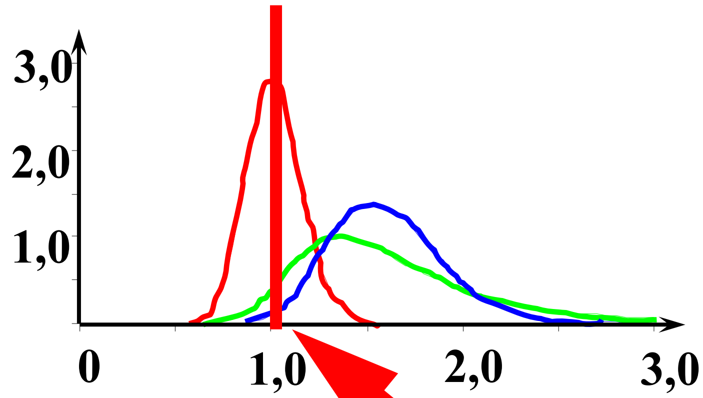
1st assumption	2nd assumption	3rd assumption
normal distr.	log. normal distr.	Gumbel distr.
estimation	estimation	estimation
$40{,}000 \times \mu_X, \sigma_X^2$	$40{,}000 \times \mu_U, \sigma_U^2$	$40{,}000 \times a, b$
computation	computation	computation
$40{,}000 \times E[X], Var[X], q_\alpha$	$40{,}000 \times E[X], Var[X], q_\alpha$	$40{,}000 \times E[X], Var[X], q_\alpha$
approximation	approximation	approximation
distributions of $E[X], Var[X], q_\alpha$	distributions of $E[X], Var[X], q_\alpha$	distributions of $E[X], Var[X], q_\alpha$

Sample Simulation (4)

Distribution of mean values



Distribution of variances



exact mean value

normal distribution

logarithnic normal distribution

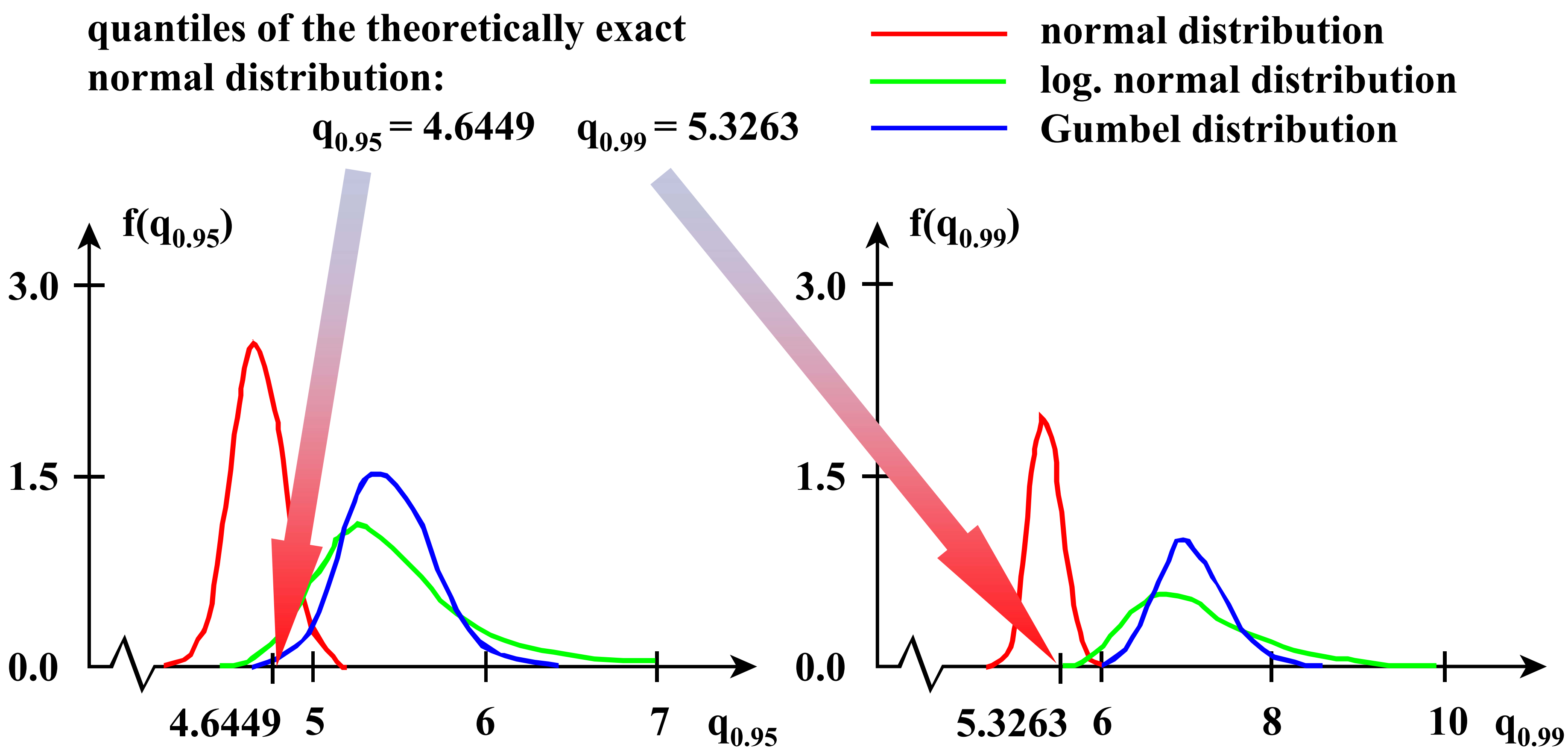
Gumbel distribution

exact variance

result: assumed type of distribution function influences the uncertain description

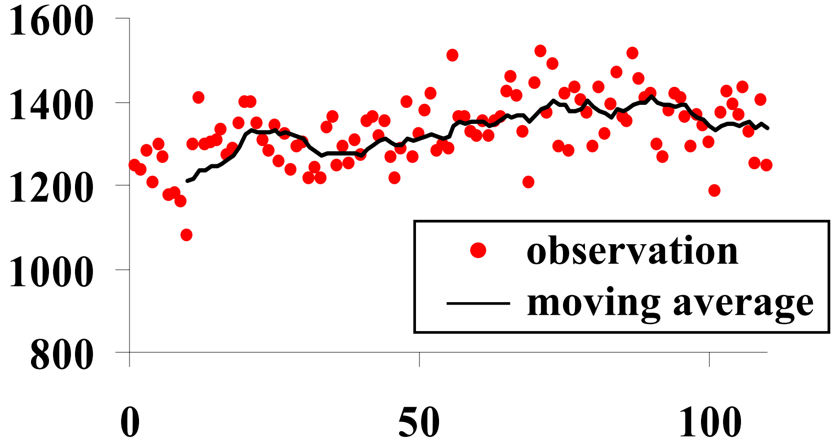
Sample Simulation (2)

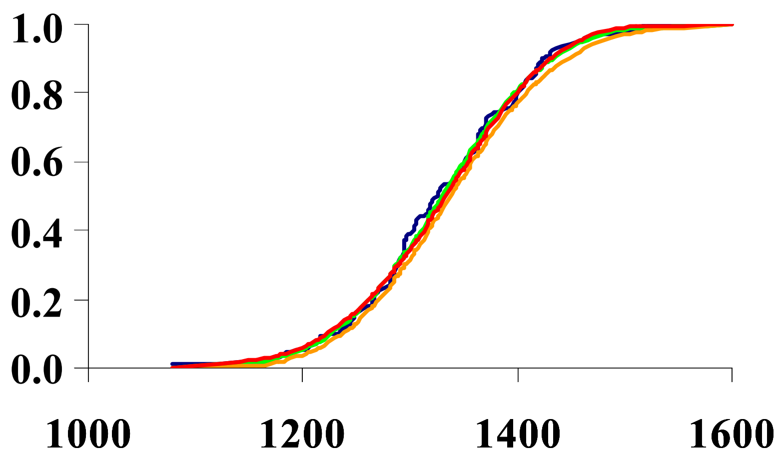
distribution of the quantiles



Test a sample of randomness

tensile strenght of glass filament yarn NEG-ARG 620-01





results of non-parametric tests

significance level

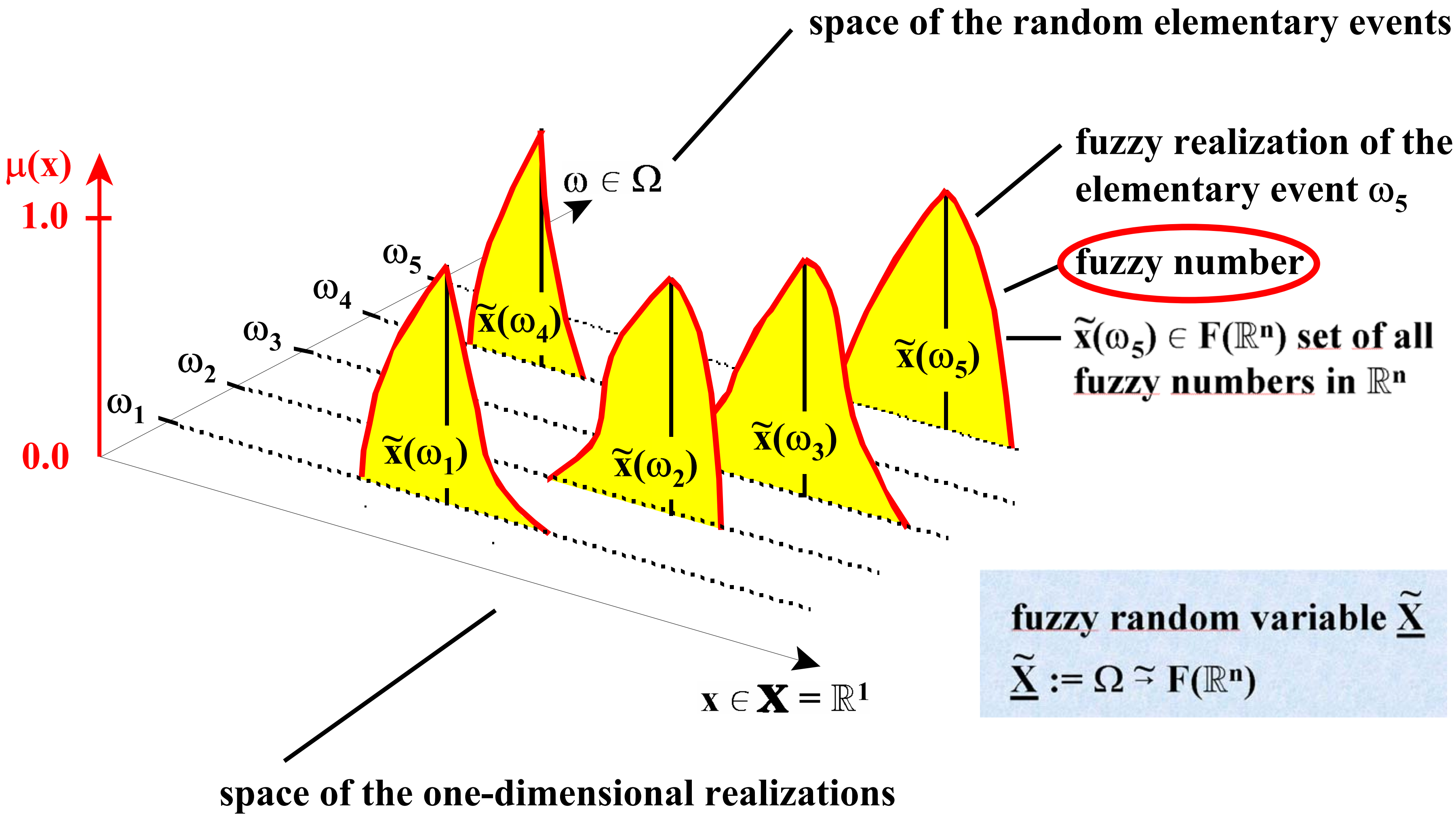
distribution function	test	1-α
	run test	0.987
	test of homogeneity	1.000
normal distribution	KS test	0.000
	χ^2 test	0.861
logarithmic normal distribution	KS test	0.000
	χ^2 test	0.721
Gumbel distribution	KS test	0.168
	χ^2 test	0.126
3-parametric Weibull distribution	KS test	0.805
	χ^2 test	0.906
2-parametric Weibull distribution	KS test	0.760
	χ^2 test	0.753

good-of-fitness tests: no unique assessment !!

Conclusion:

Not in all cases sample possesses unique determinable random properties.

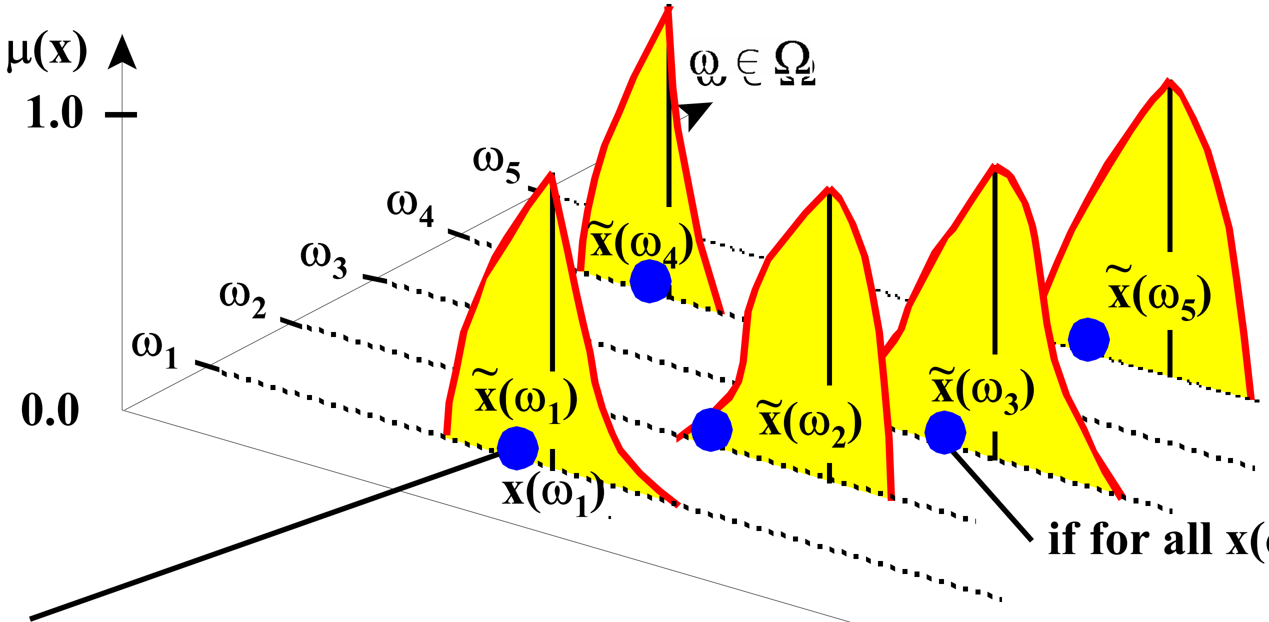
Fuzzy random variables (1)



fuzzy random variable $\tilde{\underline{X}}$

$\tilde{\underline{X}} := \Omega \rightarrow F(\mathbb{R}^n)$

Fuzzy random variables (2)



if for all $x(\omega_i)$ holds: $x(\omega_i) \in \tilde{x}(\omega_i)$
then: $\underline{x}$ constitute
an original of $\tilde{X}$

$x(\omega_1)$: realization of an ordinary
random variable X
$x(\omega_1) \in \tilde{x}(\omega_1)$

$x \in \mathbf{X} = \mathbb{R}^1$

Each original constitute
an ordinary random variable $\underline{X}$

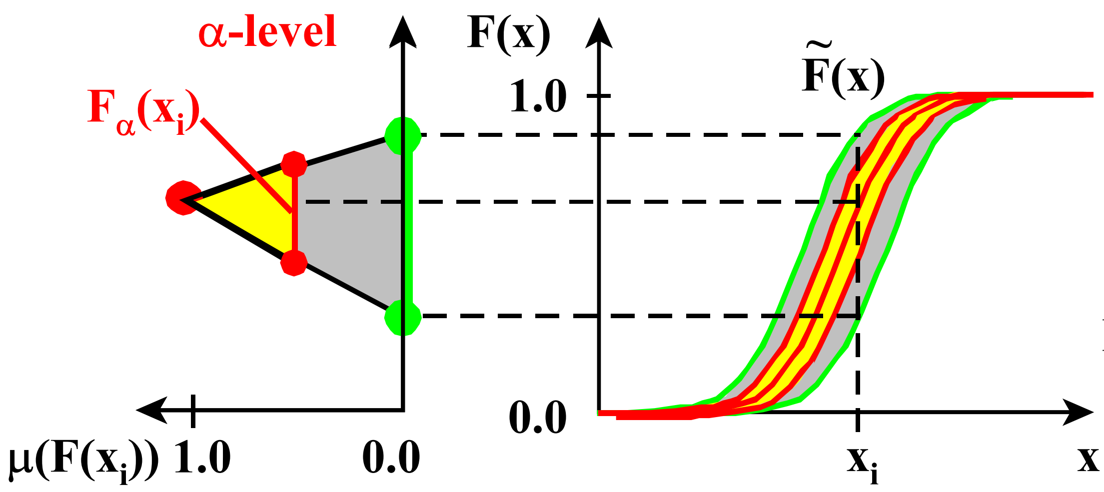
$\underline{\tilde{X}}$:= fuzzy set of all originals

originals may be described by
probalitity distribution functions

Fuzzy random variables (3)

e.g. Gumbel: $\tilde{F}(x) = \exp(-\exp(-\tilde{s}_1 \cdot (x - \tilde{s}_2)))$

$\tilde{s}_1 = \frac{\pi}{\tilde{\sigma}_X \cdot \sqrt{6}}; \quad \tilde{s}_2 = \tilde{m}_X - 0.45 \cdot \tilde{\sigma}_X$



$\tilde{F}(x) = F(\underline{\tilde{s}}, x)$

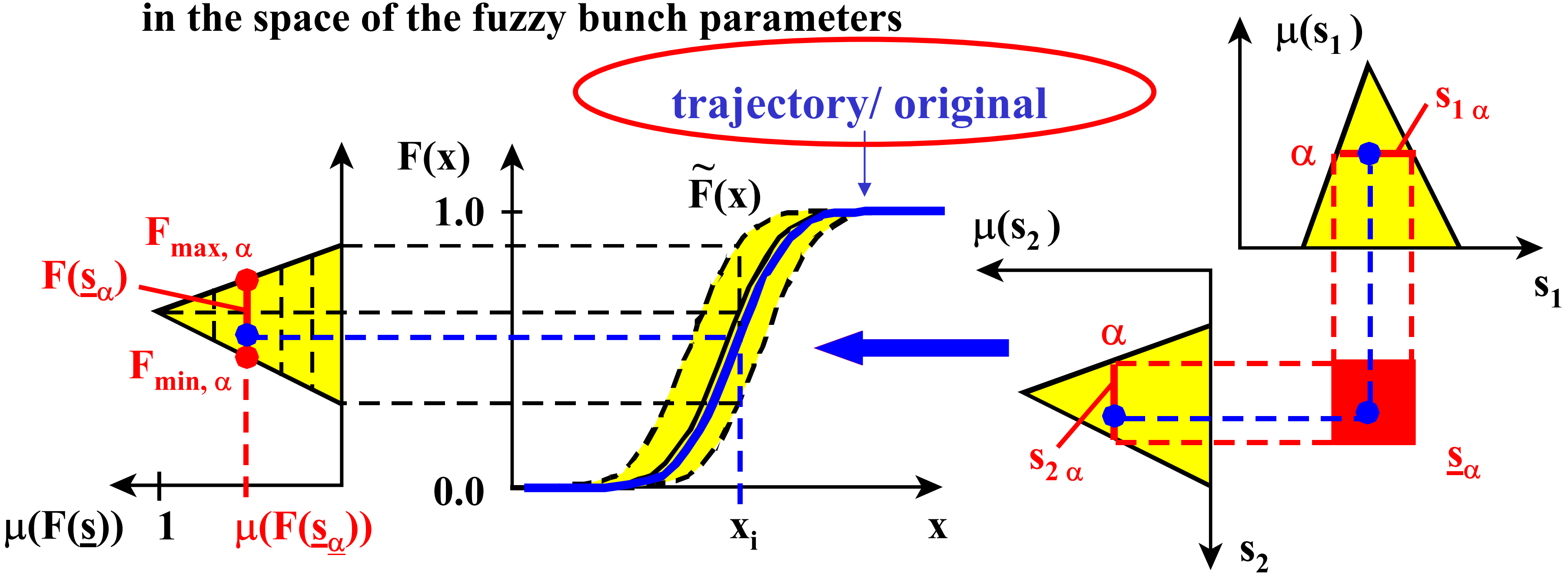
fuzzy bunch parameter vector

bunch of probability distribution functions =

fuzzy probability distribution function $\tilde{F}(x)$

Fuzzy random variables (4)

numerical handling with α-discretization in the space of the fuzzy bunch parameters



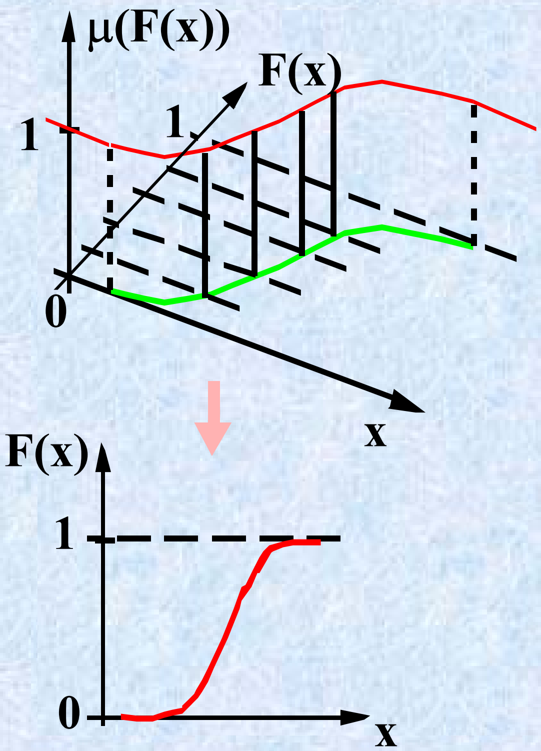
$$F(\underline{\tilde{s}},\underline{x}) = \left\{ F(\underline{s}_\alpha,\underline{x}), \mu\left(F(\underline{s}_\alpha,\underline{x})\right) \;\middle|\; \begin{array}{l} F(\underline{s}_\alpha,\underline{x}) = \left[\inf\left(F(\underline{s}_\alpha,\underline{x})\right); \sup\left(F(\underline{s}_\alpha,\underline{x})\right)\right], \\ \mu\left(F(\underline{s}_\alpha,\underline{x})\right) = \mu(\underline{s}_\alpha) = \alpha \;\forall\; \alpha \in (0;1] \end{array} \right\}$$

Uncertainty Models

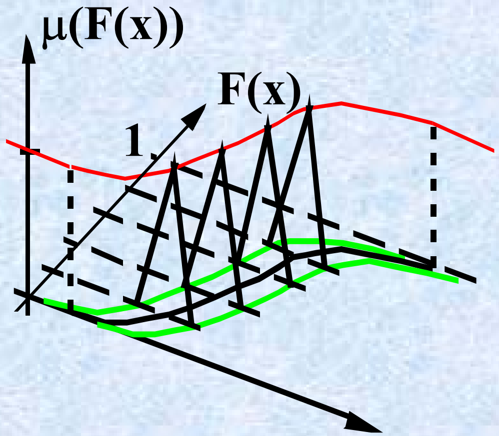
Randomness | **Fuzzy Randomness** | **Fuzziness**

special case

random variable X

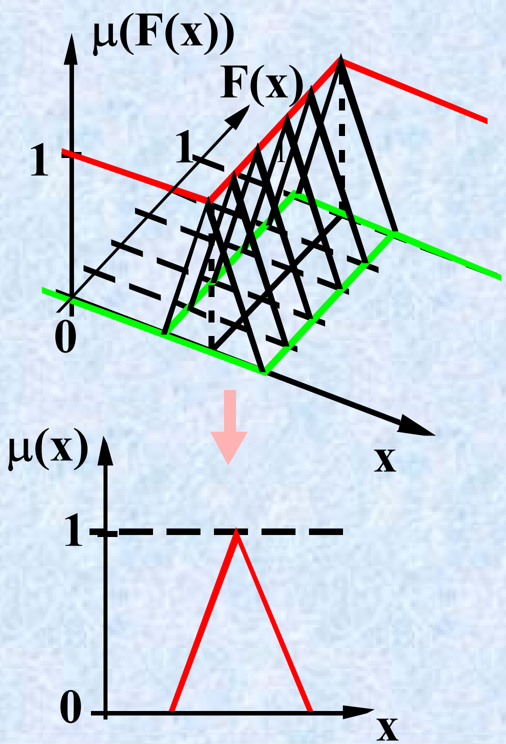


fuzzy random variable $\tilde{X}$



special case

fuzzy variable $\tilde{x}$



super ordinate uncertainty model

Fuzzy random functions (1)

given: set of fuzzy random variables $\tilde{X}(\underline{t}, \omega)$
with $\underline{t} = \{\tau, \underline{\theta}\}$, τ time, $\underline{\theta} = \{\theta_1, \theta_2, \theta_3\}$ spatial coordinates

Definition: **A fuzzy random function $\tilde{X}(\underline{t})$ is the set of fuzzy random variables $\tilde{X}(\underline{t}, \omega)$**

$$\tilde{X}(\underline{t}) = \left\{ \tilde{X}(\underline{t}, \omega) \,\middle|\, \underline{t} \in T, \omega \in \Omega \right\}$$

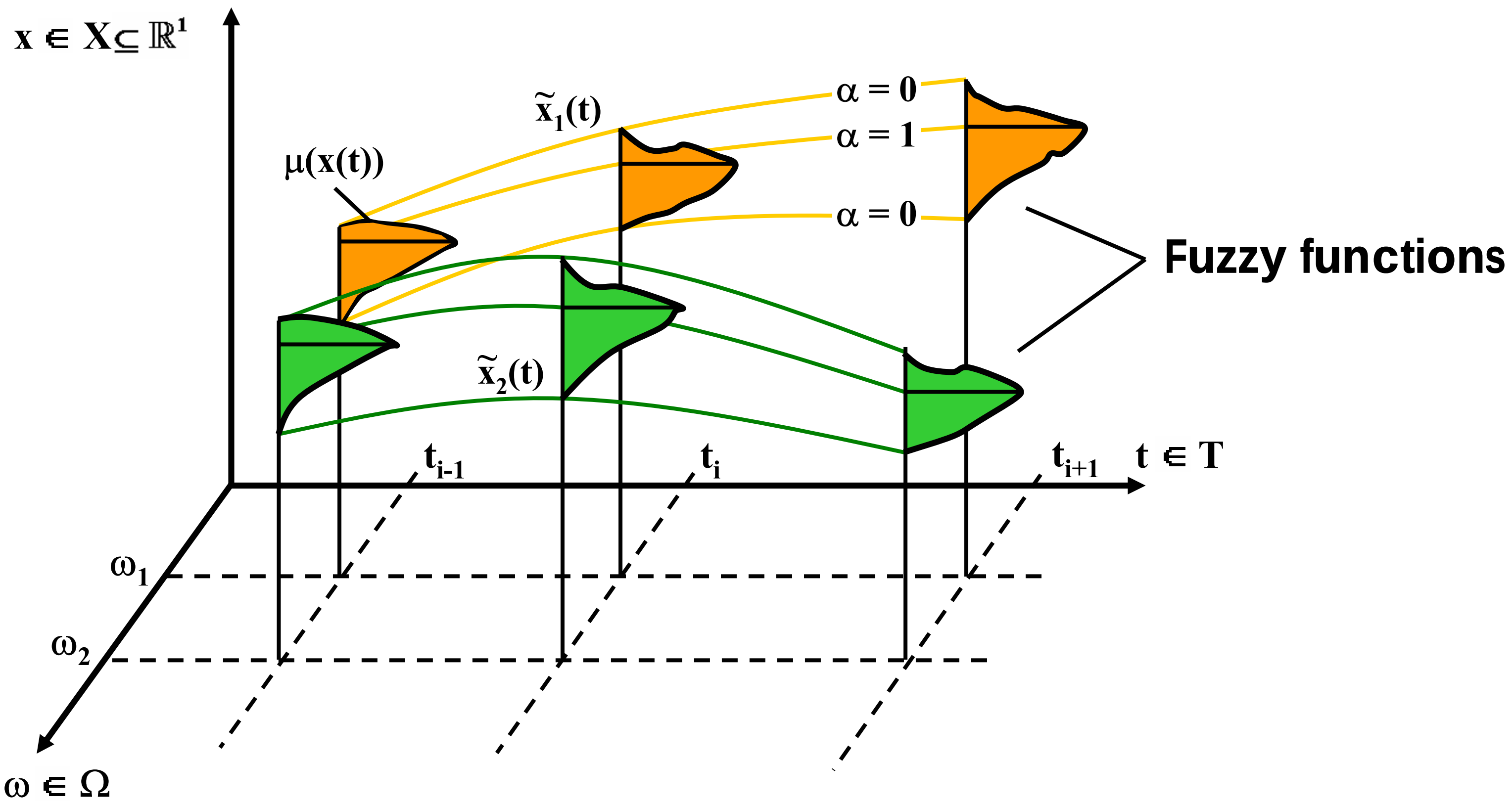
$$\tilde{X}(\underline{t}) := T \times \Omega \rightarrow F(\mathbb{R}^n)$$

special cases:

1. **no randomness:** $\tilde{X}(\underline{t}) \rightarrow \tilde{x}(\underline{t})$ **fuzzy function**
2. **no fuzziness:** $\tilde{X}(\underline{t}) \rightarrow X(\omega)$ **random function**
3. **for fixed τ :** $\tilde{X}(\underline{t}) \rightarrow \tilde{X}(\underline{\theta})$ **fuzzy random field**

Fuzzy random functions (2)

Realizations of a one-dimensional fuzzy random function



Fuzzy random fields (1)

fuzzy random field

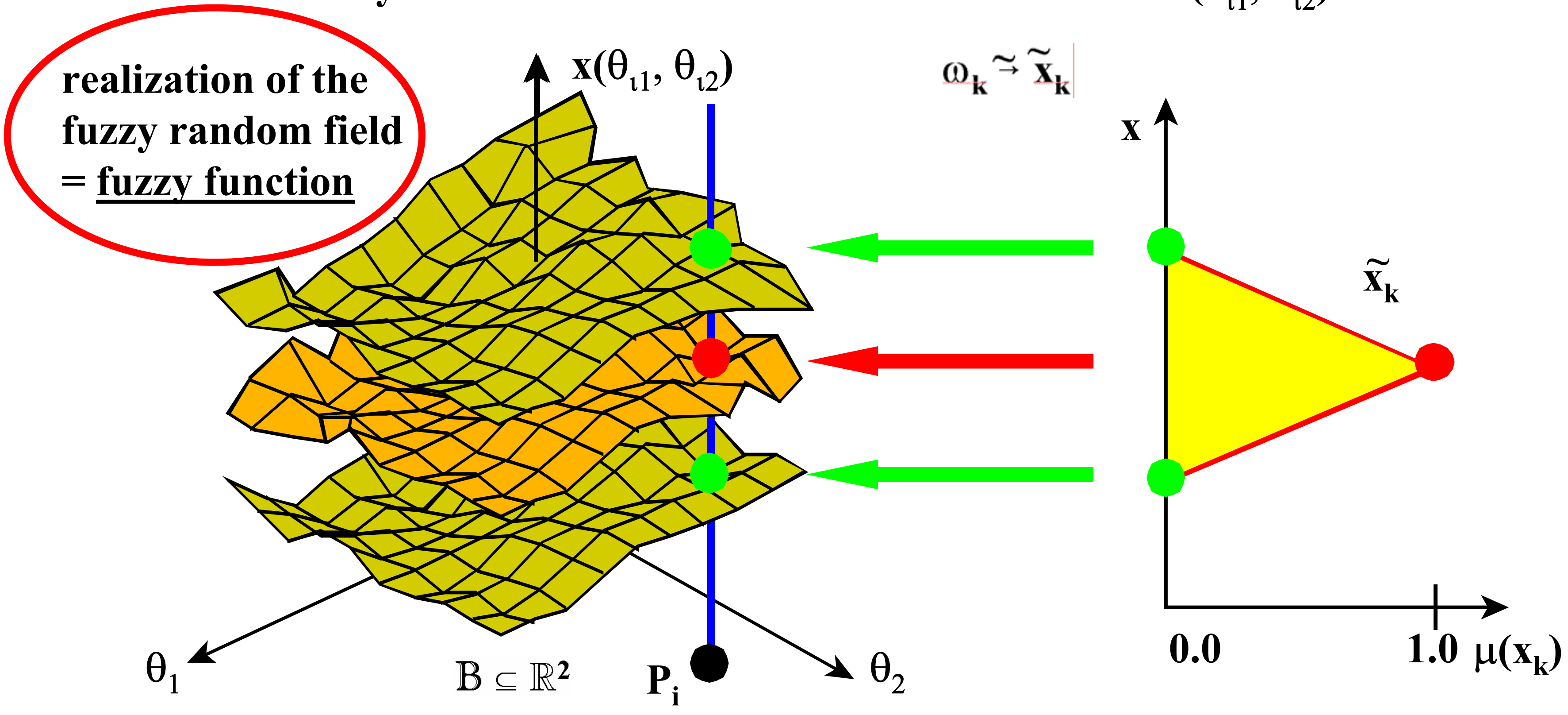
$$\tilde{X}(\underline{\theta}) = \left\{ \tilde{X}(\underline{\theta}_i) \mid \underline{\theta}_i \in B \subseteq \mathbb{R}^n \right\}$$

fuzzy random variable

fuzzy random variable in $P_i(\theta_{\iota 1}, \theta_{\iota 2})$

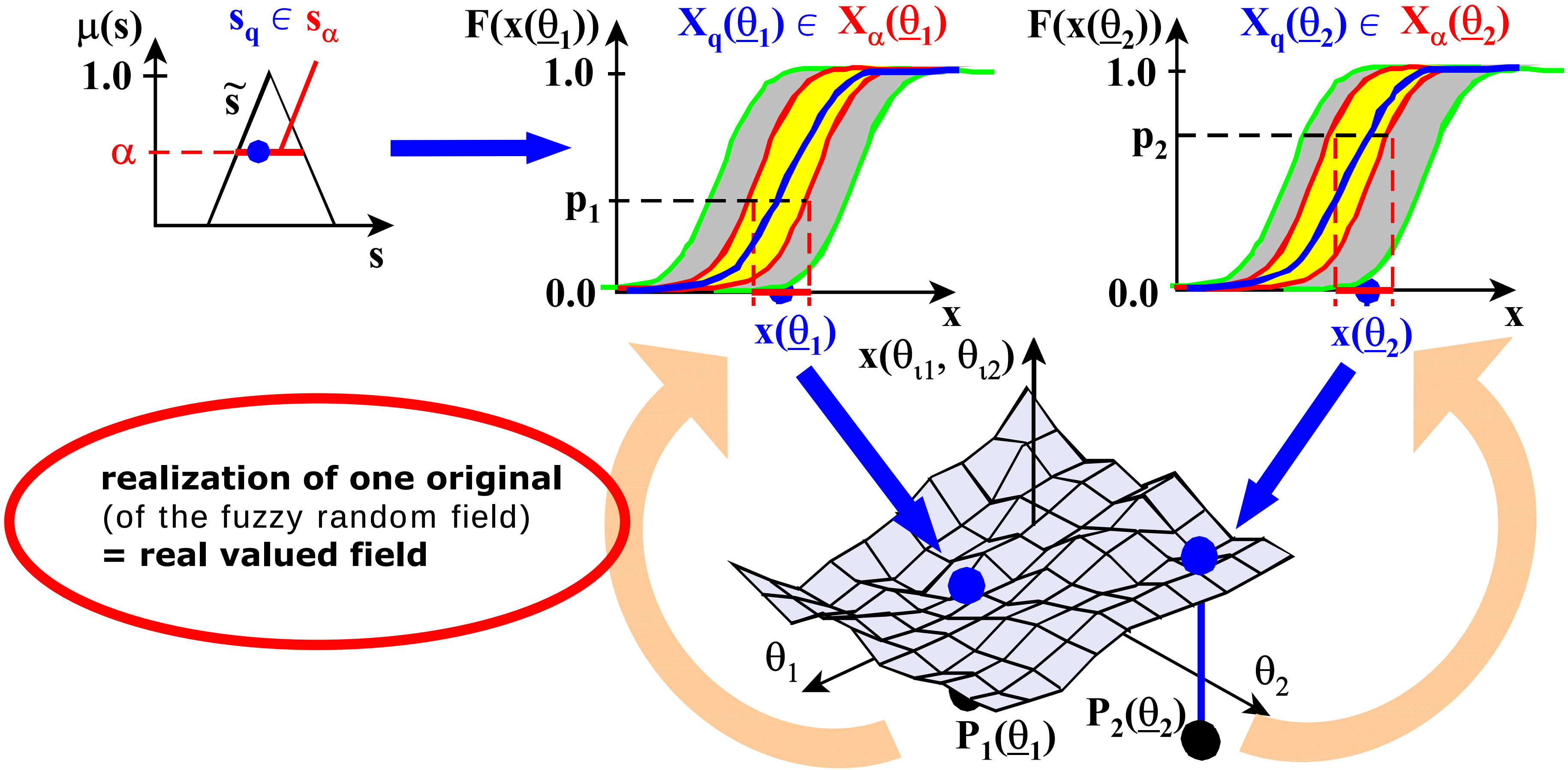
$$\tilde{X}(\theta_{\iota 1}, \theta_{\iota 2}): \Omega \to F(\mathbb{R}^1) = \left\{ \tilde{x} \mid \tilde{x} \in \mathbb{R}^1 \right\}$$

realization of $\tilde{X}(\theta_{\iota 1}, \theta_{\iota 2})$



Fuzzy random fields (2)

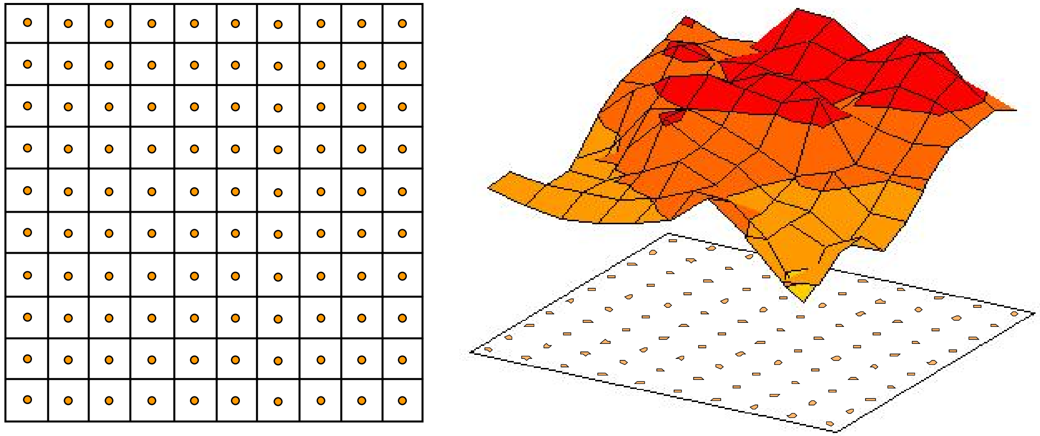
representation with fuzzy bunch parameters $\underline{\tilde{s}}$: $\tilde{X}(\underline{\theta}) = X(\underline{\tilde{s}}, \underline{\theta})$



Fuzzy random fields (3)

point discretization of fuzzy random fields

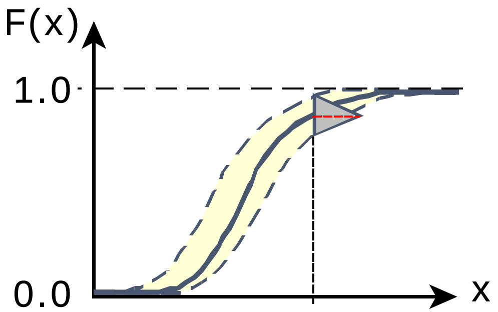
e.g. midpoint method

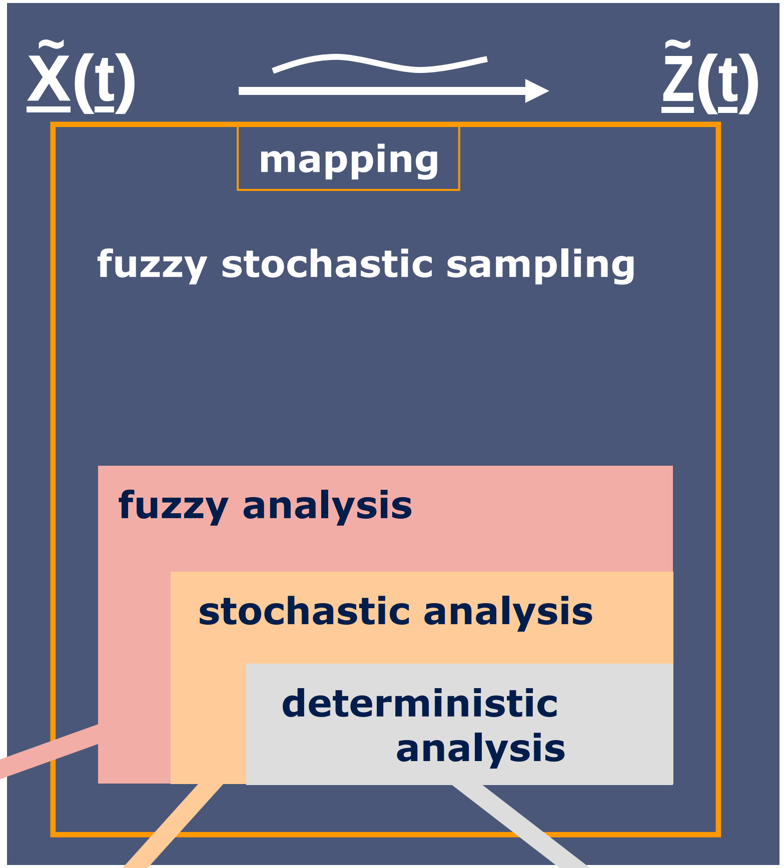


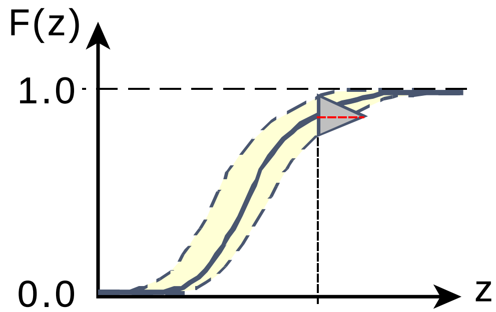
result: set of fuzzy random variables in $\mathbb{R}^n$

Fuzzy Stochastic Sampling (1)

Idea







fuzzy random

random

fuzzy

deterministic

uncertain results

α-level optimization

Monte-Carlo simulation
…

algorithmically or numerically formulated mathematical model

Space of fuzzy bunch parameters

Fuzzy random functions

$$\tilde{F}_{\theta_i}(\underline{x}) = \tilde{F}(\underline{x}, \underline{\theta}_i) = F(\underline{\tilde{s}}_i, \underline{x}, \underline{\theta}_i)$$

$$i = 1, \ldots, p_1$$

$$\rightarrow \begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_1} \end{bmatrix} \rightarrow \underline{\tilde{s}} = \begin{bmatrix} \tilde{s}_1 \\ \vdots \\ \tilde{s}_{n_1} \\ \\ \\ \\ \end{bmatrix}$$

Space of fuzzy bunch parameters

Fuzzy random functions

$$\tilde{F}_{\theta_i}(\underline{x}) = \tilde{F}(\underline{x}, \underline{\theta}_i) = F(\underline{\tilde{s}}_i, \underline{x}, \underline{\theta}_i)$$

$$i = 1, \ldots, p_1$$

Fuzzy functions

$$\underline{\tilde{x}}(\underline{\theta}_i) = \underline{x}(\underline{\tilde{s}}_i, \underline{\theta}_i)$$

$$i = 1, \ldots, p_2$$

$$\begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_1} \end{bmatrix} \quad \begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_2} \end{bmatrix} \rightarrow \quad \underline{\tilde{s}} = \begin{bmatrix} \tilde{s}_1 \\ \vdots \\ \tilde{s}_{n_1} \\ \\ \\ \end{bmatrix}$$

Space of fuzzy bunch parameters

Fuzzy random functions

$$\tilde{F}_{\theta_i}(\underline{x}) = \tilde{F}(\underline{x}, \underline{\theta}_i) = F(\underline{\tilde{s}}_i, \underline{x}, \underline{\theta}_i)$$

$$i = 1, \ldots, p_1$$

Fuzzy functions

$$\underline{\tilde{x}}(\underline{\theta}_i) = \underline{x}(\underline{\tilde{s}}_i, \underline{\theta}_i)$$

$$i = 1, \ldots, p_2$$

Random functions

$$F_{\theta_i}(\underline{x}) = F(\underline{x}, \underline{\theta}_i) = F(\underline{s}_i, \underline{x}, \underline{\theta}_i)$$

$$i = 1, \ldots, p_3$$

$$\begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_1} \end{bmatrix} \quad \begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_2} \end{bmatrix} \quad \begin{bmatrix} \underline{s}_1 \\ \vdots \\ \underline{s}_{p_3} \end{bmatrix} \Rightarrow \underline{\tilde{s}} = \begin{bmatrix} \tilde{s}_1 \\ \vdots \\ \tilde{s}_{n_1} \\ \tilde{s}_{n_1+1} \\ \vdots \\ \tilde{s}_{n_1+n_2} \\ \\ \\ \end{bmatrix}$$

Space of fuzzy bunch parameters

Fuzzy random functions

$$\tilde{F}_{\theta_i}(\underline{x}) = \tilde{F}(\underline{x}, \underline{\theta}_i) = F(\underline{\tilde{s}}_i, \underline{x}, \underline{\theta}_i)$$

$$i = 1, \ldots, p_1$$

Fuzzy functions

$$\underline{\tilde{x}}(\underline{\theta}_i) = \underline{x}(\underline{\tilde{s}}_i, \underline{\theta}_i)$$

$$i = 1, \ldots, p_2$$

Random functions

$$F_{\theta_i}(\underline{x}) = F(\underline{x}, \underline{\theta}_i) = F(\underline{s}_i, \underline{x}, \underline{\theta}_i)$$

$$i = 1, \ldots, p_3$$

Dependencies

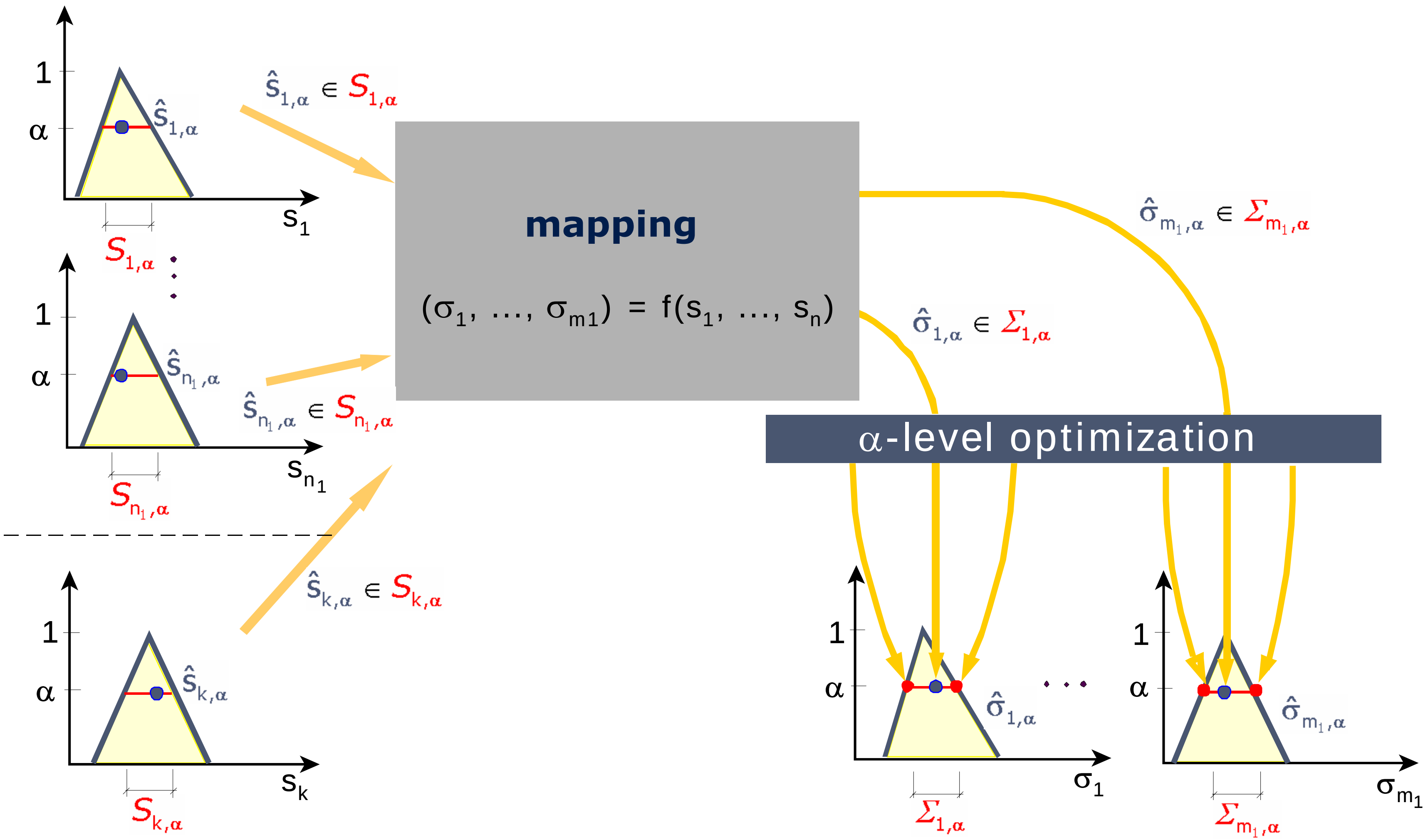
$$\tilde{k}_{x_i}(L_{12}) = k_x(\underline{\tilde{s}}_i, L_{12})$$

$$i = 1, \ldots, p_4$$

$$\begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_1} \end{bmatrix} \begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_2} \end{bmatrix} \begin{bmatrix} \underline{s}_1 \\ \vdots \\ \underline{s}_{p_3} \end{bmatrix} \begin{bmatrix} \underline{\tilde{s}}_1 \\ \vdots \\ \underline{\tilde{s}}_{p_4} \end{bmatrix} \rightarrow$$

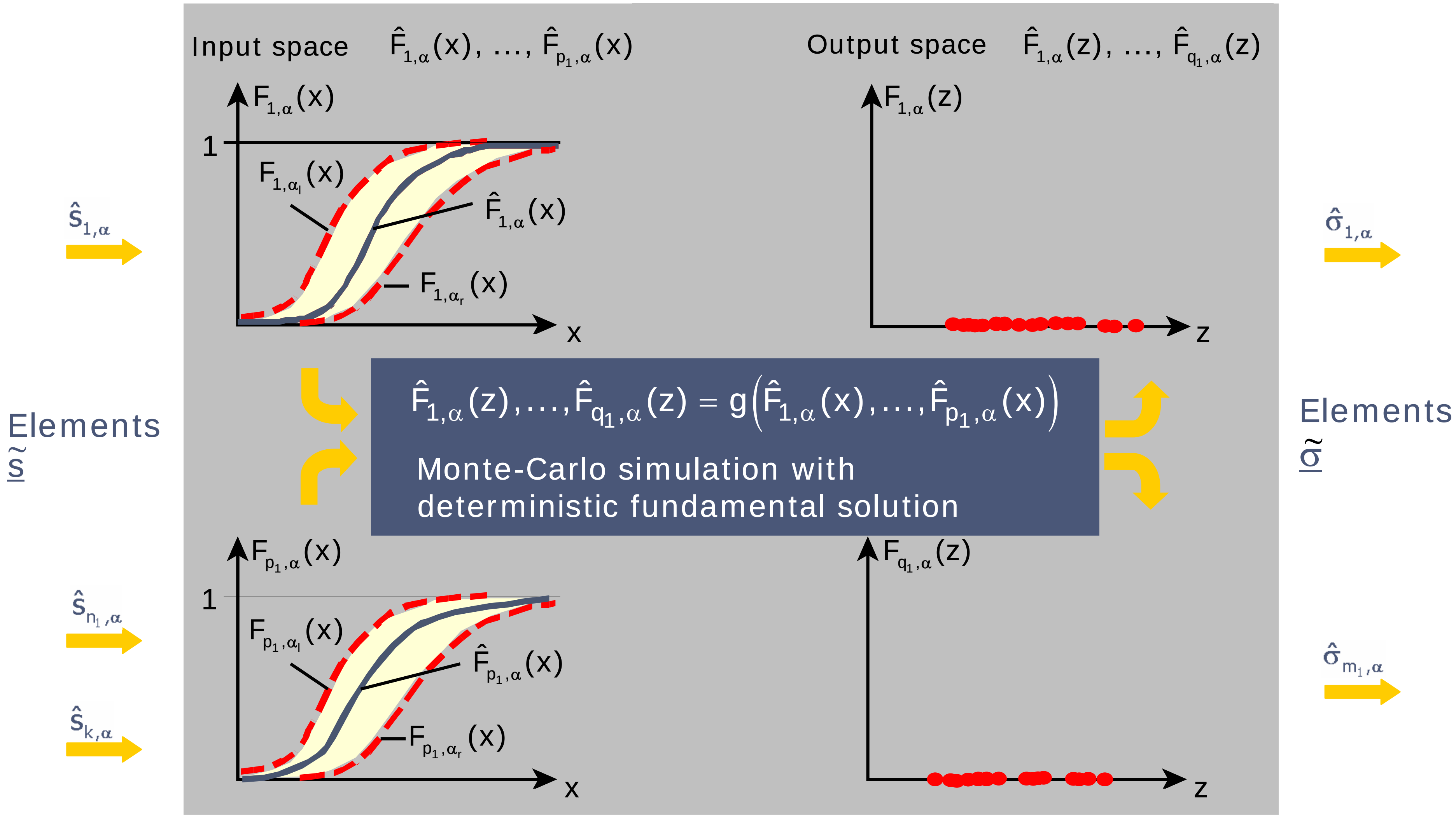
$$\underline{\tilde{s}} = \begin{bmatrix} \tilde{s}_1 \\ \vdots \\ \tilde{s}_{n_1} \\ \tilde{s}_{n_1+1} \\ \vdots \\ \tilde{s}_{n_1+n_2} \\ s_{n_1+n_2+1} \\ \vdots \\ s_{n_1+n_2+n_3} \\ \\ \end{bmatrix}$$

Fuzzy analysis in space of the fuzzy bunch parameters



Fuzzy Stochastic Sampling (4)

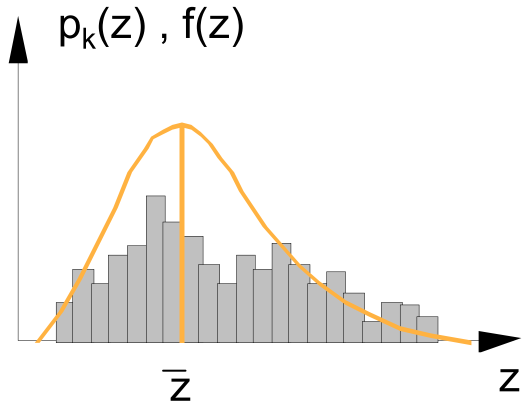
Mapping



- **mathematical samples with n elements for every result variable z**
- **estimation of parameters of the samples (expected value, variance)**
- **estimation of quantils of the samples**
- **empirical distribution function**
- **tests for different types of theoretical probability distribution functions – estimation of parameters**

Bunch parameters $\hat{\underline{\sigma}}$

Histogram



$$\bar{z} = \sum_n z_k \cdot p_k$$

$$\sigma_z = \sqrt{\sum_n (z_k - \bar{z}) \cdot p_k}$$

Fuzzy Stochastic Sampling (4)

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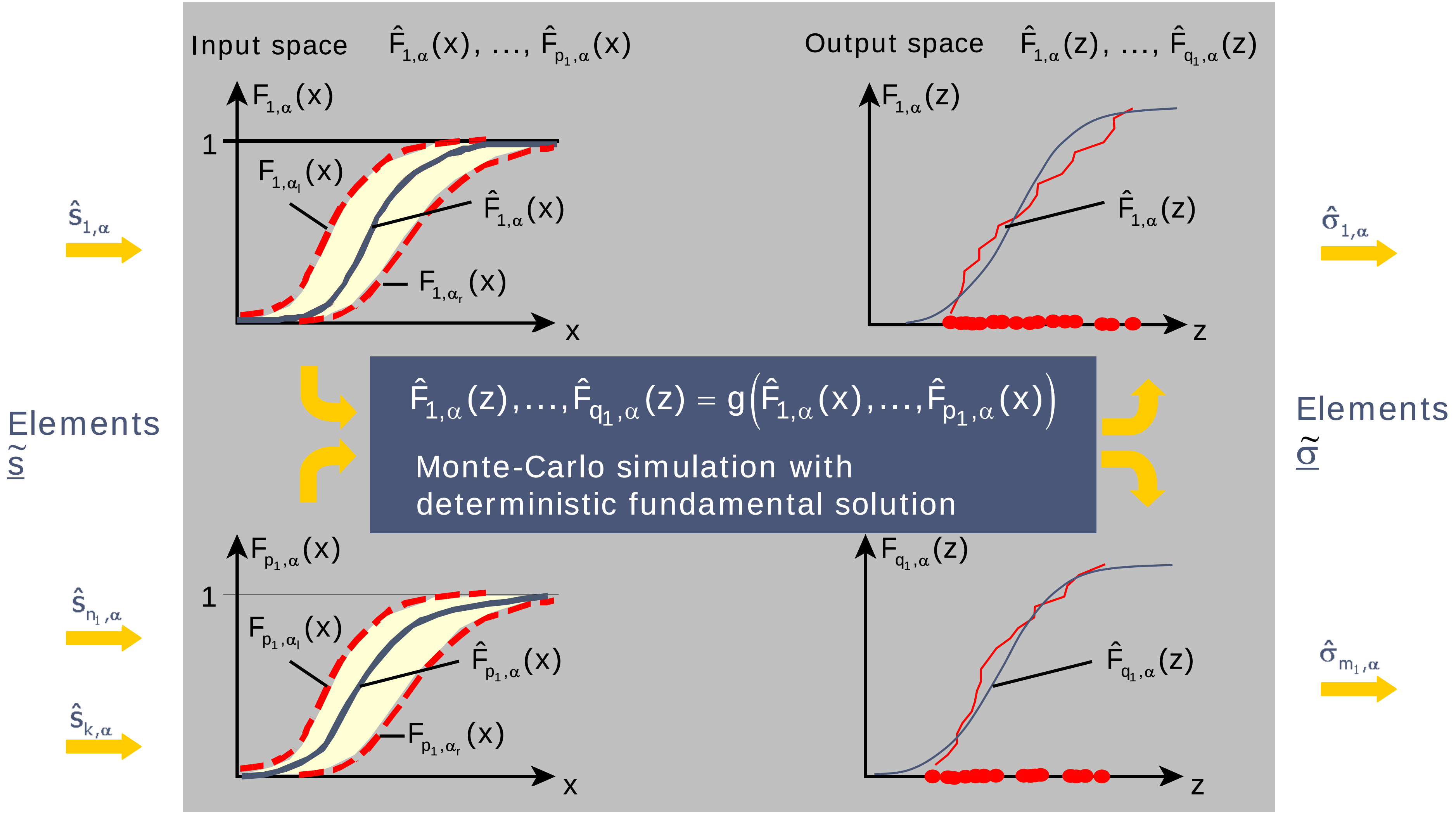
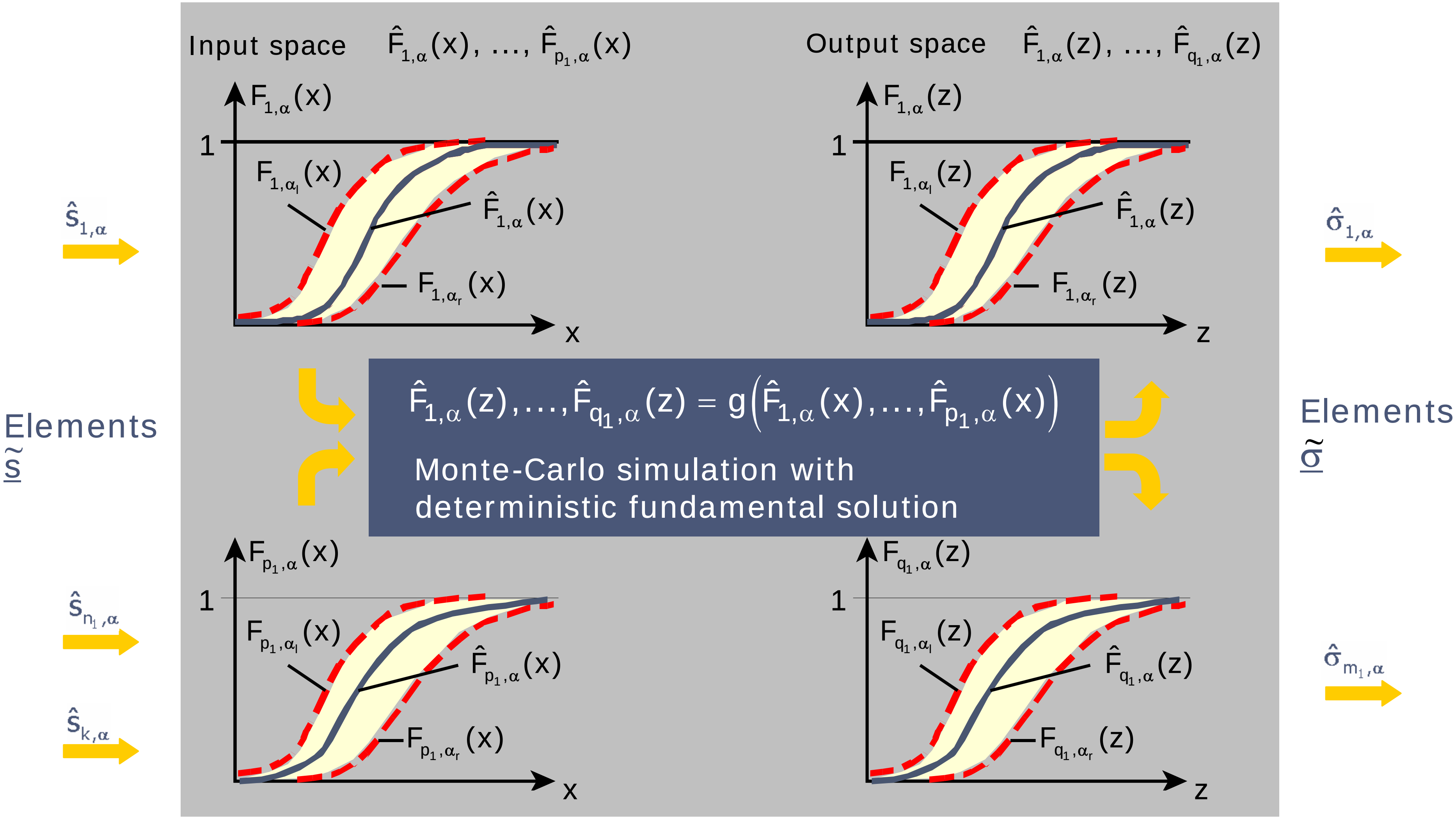
Mapping



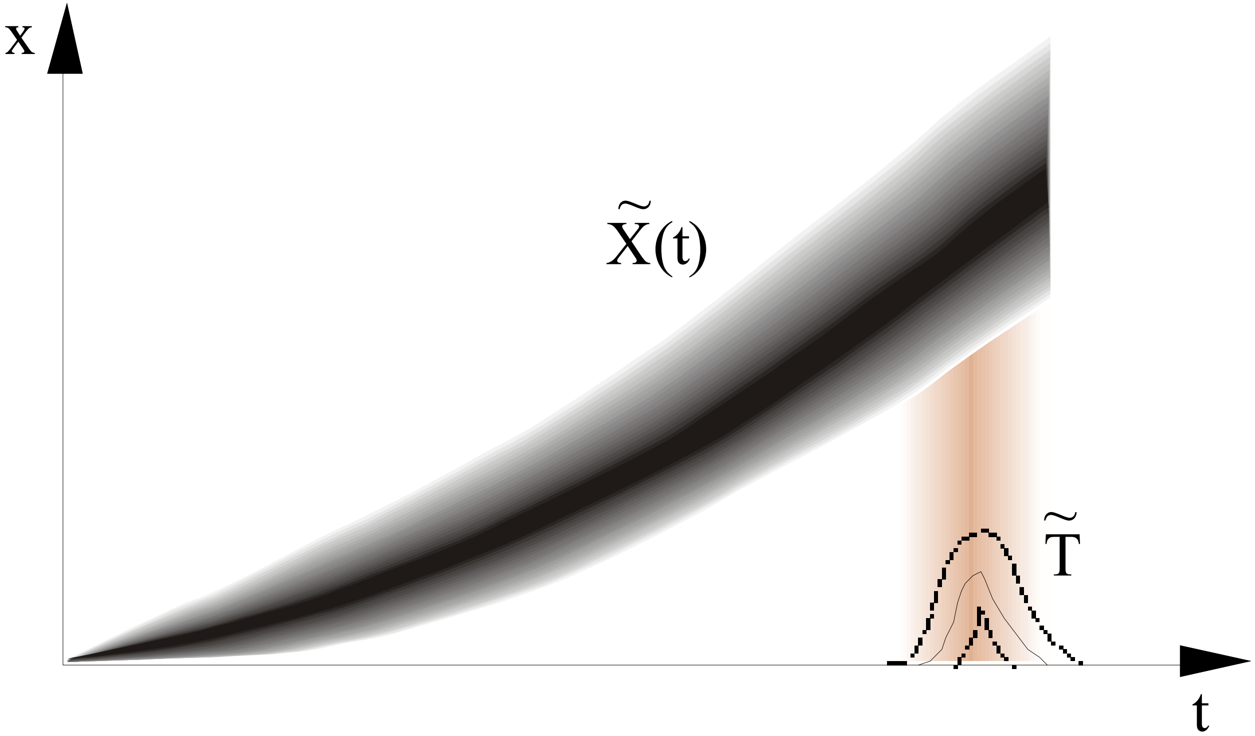
Fuzzy Stochastic Sampling (4)

Mapping

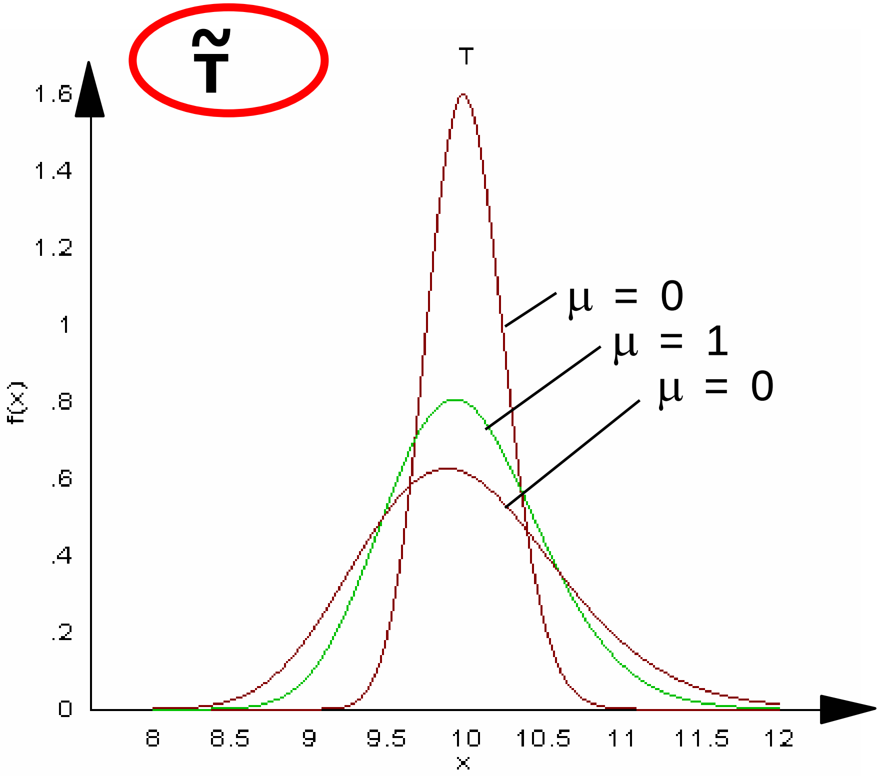


Computation of a fuzzy stochastic integral

$$\tilde{I} = \int_0^{\tilde{T}} \tilde{X}(t)\, dt = \int_0^{\tilde{T}} \left[\tilde{a} \cdot t^2 + \frac{1}{\tilde{E}} \cdot t \right] dt$$



Fuzzy random variables

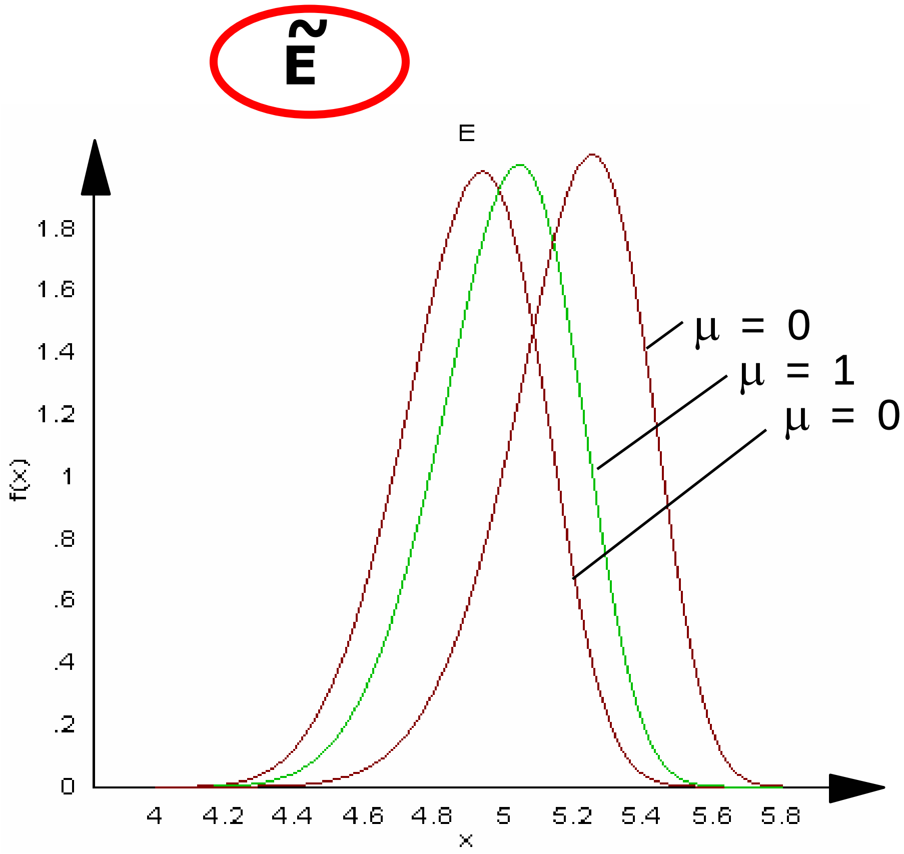


fuzzy logarithmic normal distribution

$m_T = 10$

$\tilde{\sigma}_T = <0.25, 0.5, 0.65>$

$x_0 = 5.0$



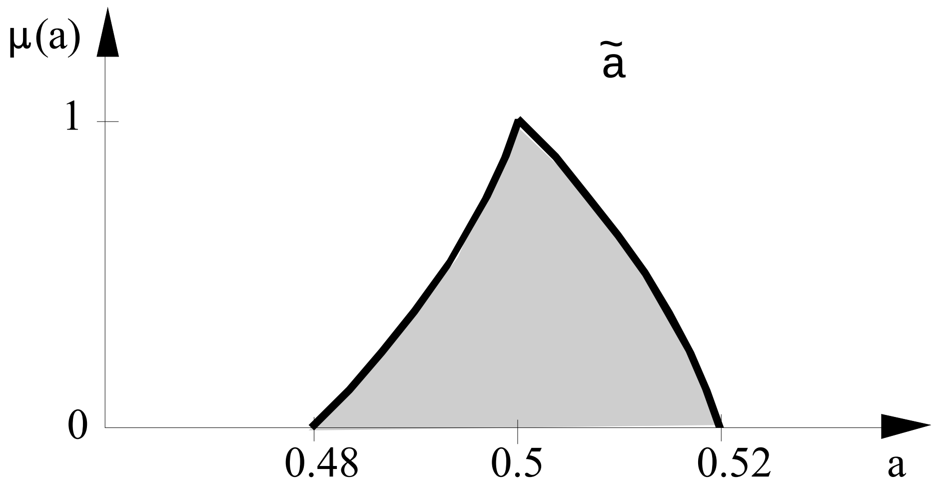
fuzzy Weibull distribution

$\tilde{m}_E = <4.9, 5.0, 5.2>$

$\sigma_T = 0.2$

$x_0 = 4.0$

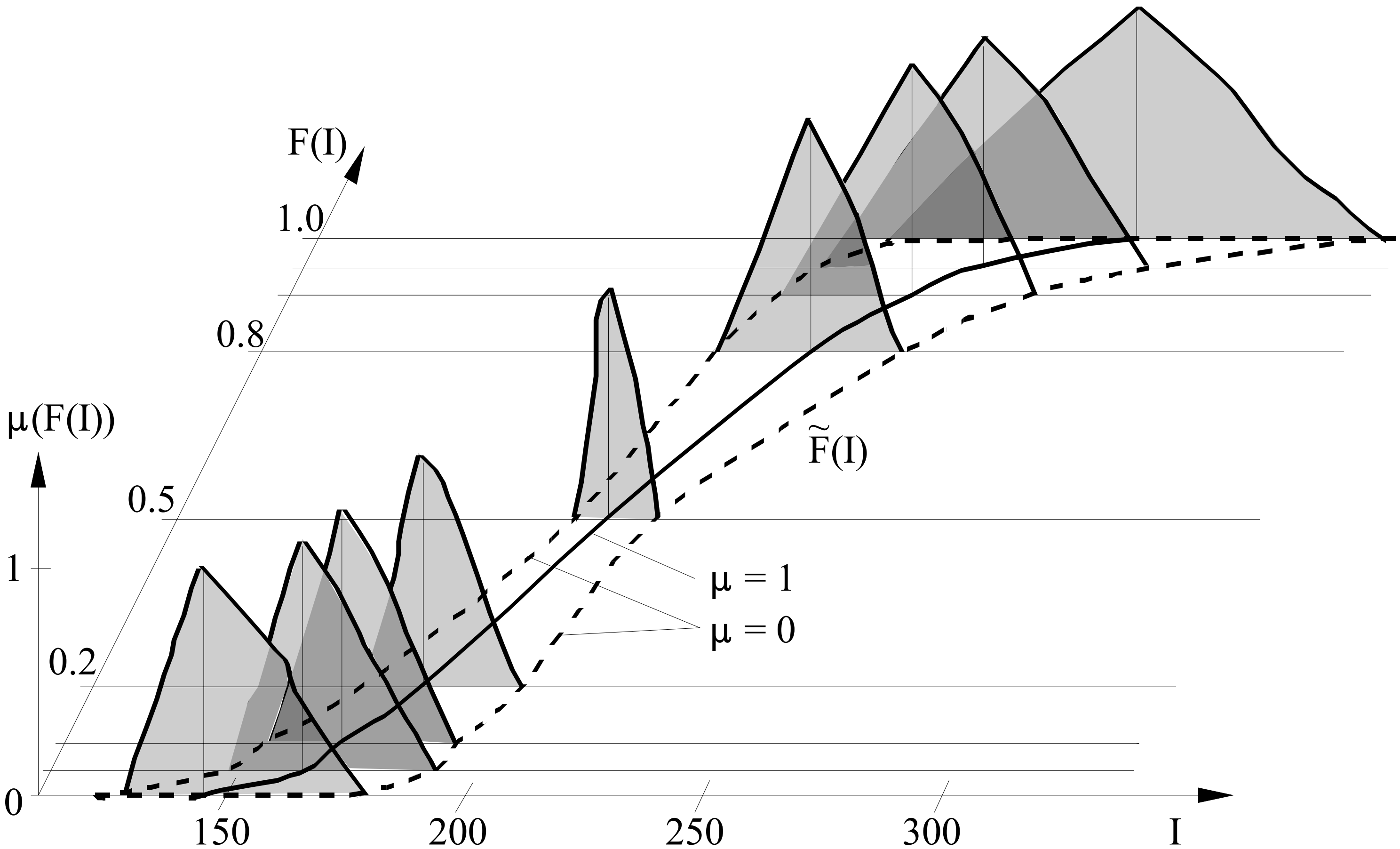
Fuzzy variables



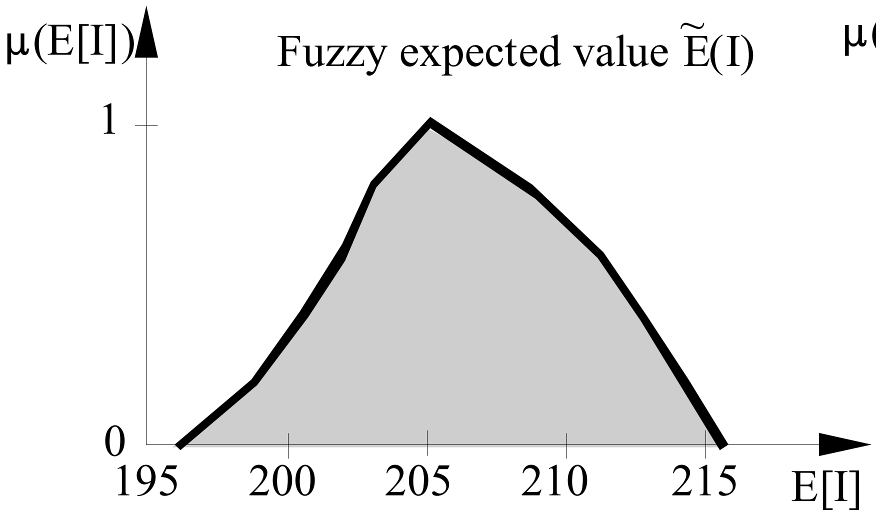
nonlinear membership function

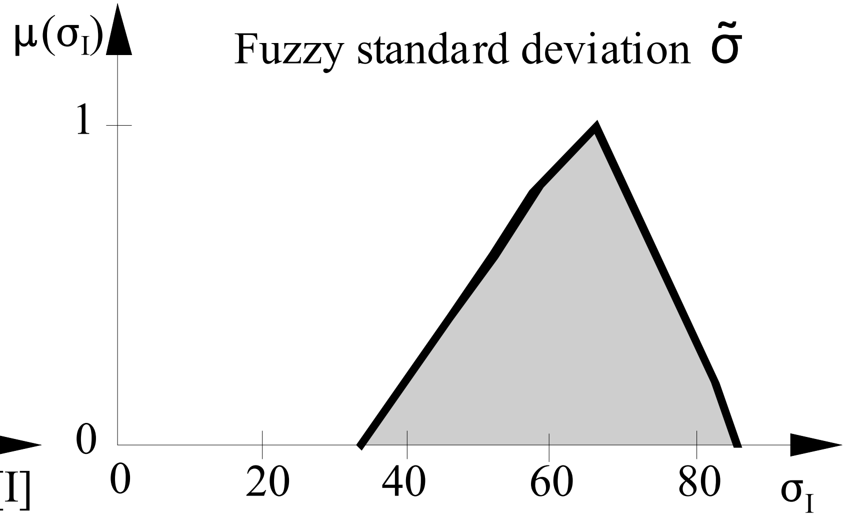
- bunch parameters are subdivided into 5 α-level
- 20 optimization steps on each α-level = number of samples
- 10 000 elements per sample → yields one trajectory

Empirical fuzzy probability distribution function



Fuzzy bunch parameter of the fuzzy stochastic integral $\tilde{I}$





Thank you !