Interval Finite Element Methods: New Directions

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Outline

1. Introduction
2. Intervals and Probabilities
3. Interval FEM and *hp*-FEM
4. Discrete Maximum Principles
5. Hermes

History of Interval Analysis

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- Situations with conflicting information (Ferson & Ginzburg, 1996)
- Fuzzy arithmetic on α-cuts (Lodwick & Jamison, 2002)

Interval Finite Element Methods

Optimization Approaches

Non-Optimization Approaches

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Optimization Approaches

- Computation of a minimum and maximum (structural response)

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- Linear problems $\rightarrow$ system of linear interval equations

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Non-Optimization Approaches

- Linear problems $\rightarrow$ system of linear interval equations
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- Moore (1979), Neumaier (1990), Hansen (1992), Muhanna & Mullen (2001)

Intervals and Probabilities

Uncertainty

Uncertain Finite Element Models

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Uncertainty

- Probabilistic
  - Monte Carlo methods
  - Analytic techniques

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Uncertain Finite Element Models

- Two extrema: either fully probabilistic, or intervals only
- Challenge: efficient combination

hp-FEM

Differences from traditional FEM

- Elements with varying h and p
- Efficient resolution of singularities and multi-scale phenomena
- Exponential rate of convergence

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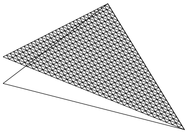
- Elements with varying *h* and *p*
- Efficient resolution of singularities and multi-scale phenomena
- Exponential rate of convergence

Challenges

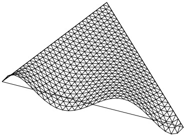
- Implementation (mainly in 3D)
- Theory
  - Design of finite elements
  - A-posteriori error estimates
  - Discrete maximum principles, etc.

Shape functions on triangular elements



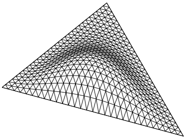
- Vertex functions

Shape functions on triangular elements



- Vertex functions
- Edge functions

Shape functions on triangular elements



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- Edge functions
- Bubble functions

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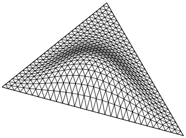


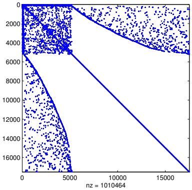
- Vertex functions
- Edge functions
- Bubble functions

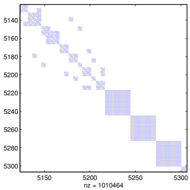
Number of shape functions

- Vertex: 3
- Edge: $(p - 1)$ on each edge
- Bubble: $(p - 1)(p - 2)/2$

Stiffness matrix

VV	VE	VB
VE	EE	EB
VB	EB	BB

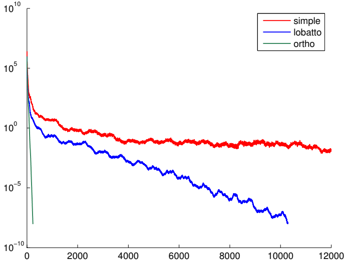




- BB-block is diagonal with suitable choice of bubble functions (symm. problems)
- Efficient computation of Schur complement

Example: Convergence of Matrix Solvers

Convergence plot of CGM

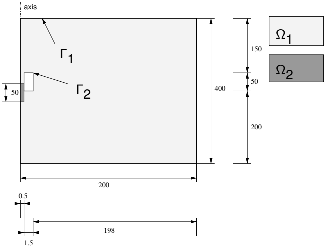


Bubble functions & Cond. number	
Monomial-based	3.3830e+009
Integr. Legendre	1.0661e+007
Orthogonal	1.3882e+003

- 10^{th} degree elements, matrix rank: 1742
- Solin, Vejchodsky (Math. Comp. Sim., to appear)

Example 1: Electrostatic Micromotor

Motors resisting destructive electromagnetic waves



(adjusted scaling)

- Electric potential
  - $\varphi = 0$ V on Γ_1
  - $\varphi = 50$ V on Γ_2
- Permittivity
  - $\epsilon = 1$ in Ω_1
  - $\epsilon = 10$ in Ω_2

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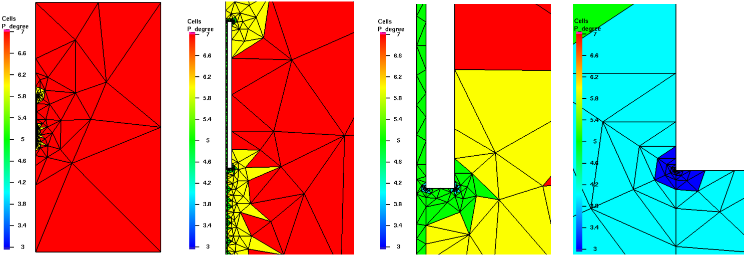

The *hp*-mesh (zoom = 1, 6, 50, 1000). Farfield: large seventh-order elements. Re-entrant corners: small lowest-order elements.

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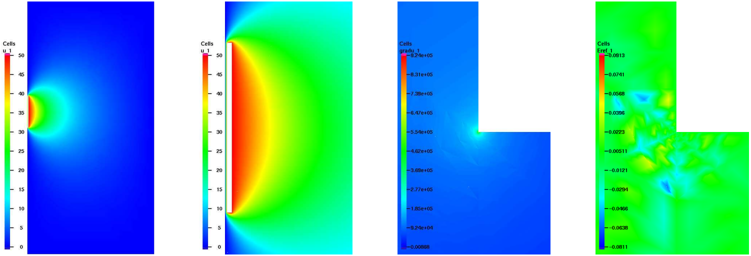
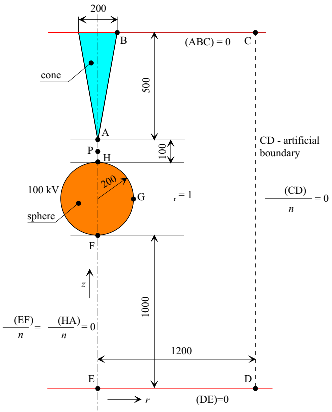


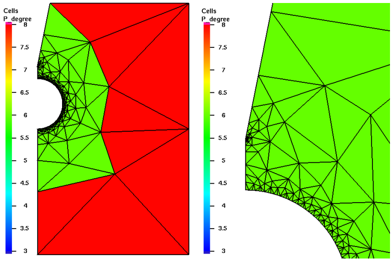
Electric potential φ (zoom = 1 and 6), singularity of $|\mathbf{E}| = |-\nabla\varphi|$ (zoom = 1000), a-posteriori error estimate (zoom = 1000).

Example 1: Electrostatic Micromotor

	Lowest-order FEM	*hp*-FEM
DOF	472384	4511
Error est.	0.2024 %	0.173 %
Iterations	387	71
CPU time	**32 min.**	**17 sec.**

Example 2: Sphere-Cone Problem





Potential on the sphere:

$\varphi = 100\ \mathrm{kV}$

hp-mesh: global view and detail

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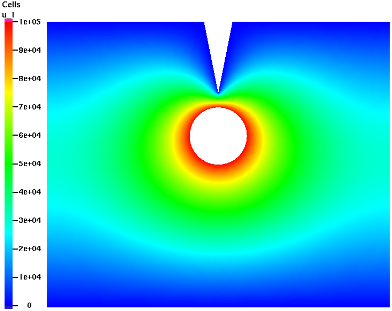
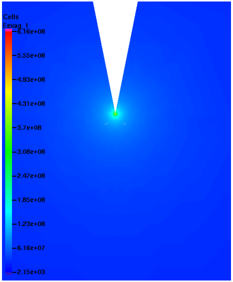


Solution to the sphere-cone problem (electric potential φ)



Detail of the singularity $|E| = |-\nabla\varphi|$ at the tip of the cone (zoom 10000)

Example 2: Sphere-Cone Problem

	Lowest-order FEM	*hp*-FEM
DOF	488542	3317
Error est.	0.5858 %	0.2804 %
Iterations	859	44
CPU time	**30 min.**	**10.53 sec.**

More examples at http://hpfem.math.utep.edu.

Interval FEM & *hp*-FEM

Benefits

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Open questions

- Properties of interval matrices
- Note: *hp* brings no new dependency issues

Discrete Maximum Principles

Classical maximum principle for elliptic problems

Let $Lu = f$, $f \leq 0$ in Ω, and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.
Then u attains its maximum on the boundary $\partial\Omega$.

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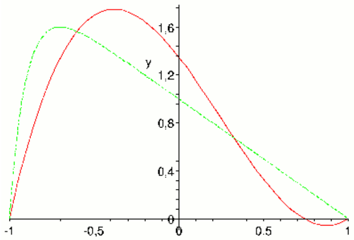
- Possible violation of physics $\rightarrow$ computational errors
- Fixing this problem: additional restrictions on mesh geometry
- Almost no results available for *hp*-FEM

One-Dimensional Example

Poisson equation $-u'' = f$ in $(-1, 1)$, $f(x) = 200e^{-10(x+1)} \geq 0$.



Exact solution (green), cubic approximation (red).

Recent Results for *hp*-FEM

Poisson equation in 1D

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- DMP holds if $f \geq 0$ and max. rel. element length is 90%.
  (T. Vejchodsky, P. Solin: Discrete Maximum Principle for the hp-FEM, 2006, submitted)

Role of Interval Analysis

Discrete Green's function (DGF)

- $G_{h,p}(\boldsymbol{x},\boldsymbol{z})$ is solution of $Lu(\boldsymbol{x}) = \delta(\boldsymbol{z})$. It satisfies

$$u_{h,p}(\boldsymbol{x}) = \int_\Omega G_{h,p}(\boldsymbol{x},\boldsymbol{z}) f(\boldsymbol{z})\, \mathrm{d}\boldsymbol{z}$$

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- DMP $\Leftrightarrow$ DGF is nonnegative in Ω^2
- Challenges:
  - DGF defined in 4D for two-dimensional problems
  - Find global minimum of DGF
  - Verify nonnegativity of DGF in certain regions
  - Intervals proved helpful (P. Solin, T. Vejchodsky, R. Araiza: Conservation of Nonnegativity for Elliptic Problems in 1D Solved by hp-FEM, Math. Comp. Sim., to appear)

The HERMES Project: *Hierarchic Modular hp-FEM System*[1]

Nonlinear coupled problems

Elasticity, heat transfer, electromagnetics, viscous flow

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Approximation of physical fields

- Various types of *hp* finite elements (continuous, edge, T-H)
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Software aspects

- Built upon PETSc: parallel and modular architecture
- Home page: http://hpfem.math.utep.edu

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Invitation to FEMTEC

FEMTEC 2006 homepage:
http://servac.math.utep.edu/femtec_2006

End of Talk

Thank You!