# Boundary Element Analysis of Systems Using Interval Methods

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**Abstract:** In engineering, most governing partial differential equations of physical systems are solved using finite element or finite difference methods. Applications of interval methods have been explored in finite element analysis to model systems with uncertainty in parameters and to account for the impact truncation error on solutions. An alternative to finite element analysis is boundary element method. The boundary element method uses singular functions to reduce the dimension of the domain by transforming the domain variables to variables on the boundaries. In this work, new methods using interval variables are developed to enhance boundary element method for considering impreciseness such as uncertain boundary conditions, truncation errors, integration errors and discretization errors. Exemplars are presented to illustrate the effectiveness and potential of interval approach in boundary element method analysis.

**Keyword:** boundary element method, interval analysis, truncation error, discretization errors

## 1. Introduction

Boundary element analysis (BEA) is a method for obtaining approximate solution of partial differential equations. This method requires less meshing than finite element analysis and thus, it is comparatively faster in generating or refining the mesh. BEA is performed by transformation of the domain variables to the variables on the boundaries of the system. The domain transformation is constructed using singular solutions of the governing partial differential equation. Though extensions to non-linear problems can be of the domain, straight forward BEA formulations apply to linear problems. Then the transformed boundary integral equations are solved using collocation methods, i.e., source points are located sequentially at all boundary nodes that map the domain variables such that they coincide to their values at the nodes.

Errors in BEA can be classified into the following sources:

1) Uncertainty in the boundary conditions
2) Uncertainty in parameters of the system
3) Errors in integration
4) Errors in the solution of the resulting linear system of equations
5) Discretization errors.

In this paper we will address the use of concepts from interval methods to address all of the above except for the issue of uncertainty in system parameters. If system parameters such as material properties change, one may need to develop a new analytical singular solution. When the boundary conditions are uncertain, the use of intervals to bound this uncertainty leads to a system of linear equations with an interval right hand side. The incorporation of this source of uncertainty can be treated is a manner similar to that used in finite element analysis (Mullen and Muhanna, 1999).

Most boundary element programs use numerical quadrature to integrate terms in the resulting system of linear equations. In some problems, one can perform the integration explicitly; other BEA may require integration that may not be generally performed explicitly. One procedure to overcome this issue is to expand the mapping functions as a series, such as Taylor series expansion. This expansion, in fact, is an approximation of the function in the form of a polynomial, using the function's derivatives evaluated at a point inside the domain of the function. The truncation error is considered as an interval variable obtained from the maximum Taylor series expansion remainder. Then, the BEA is performed in the presence of variation in the corresponding linear system of equations. Based on present error bounds, the enclosure on the bounds of the results is quantified. This procedure can lead to interval bounds on errors due to integration.

Truncation errors in the solution of the resulting system of linear equations can be included in BEA using conventional interval methods for linear equations (Alefeld 1983, Gay 1982, Hansen 1965, Jansson 1991, Moore 1979, Neumaier 1987, 1988, 1990, Rump 1990, Sunaga 1958).

Finally we explore the bounding of discretization errors using local functions that are bounded by interval values. An example for a two dimensional Laplace equation using constant elements is presented. Sharp bounds require a method for solving parametrically constrained systems of linear equations.

## 2. Boundary Element Analysis of Laplace Equation

### 2.1. BEA FORMULATION FOR LAPLACE EQUATION

The theory of boundary elements is discussed in the books by Brebbia 1992 and Hartmann 1889. In the following, we will review a two dimension boundary element formulation for Laplace equation.

The Laplace equation is:

$$\begin{aligned} \nabla^2 u = 0 \quad & in\ \Omega \\ u = \hat{u} \quad & on\ \Gamma_1 \\ \frac{\partial u}{\partial n} = q = \hat{q} \quad & on\ \Gamma_2 \end{aligned} \tag{1}$$

where $(\Omega)$ is the domain of the system, $(\Gamma)$ is the boundary of the system and $(\hat{u})$ and $(\hat{q})$ are the values at the boundary.

To minimize the error introduced as the exact solution of $(u)$ and $(q)$ is approximated, orthogonalization of Eq. (1) with respect to a test function $(w)$ is performed:

$$\int_{\Omega} \nabla^2 u w d\Omega = \int_{\Gamma_2} (q - \hat{q}) w d\Gamma - \int_{\Gamma_1} (u - \hat{u}) \frac{\partial w}{\partial n} d\Gamma \tag{2}$$

Twice integrating by parts on the left side of Eq. (2) and considering $u^* = w$ and $q^* = \partial u^* / \partial n$ yields:

$$\int_{\Omega} \nabla^2 u^* u d\Omega = -\int_{\Gamma_2} \hat{q} u^* d\Gamma - \int_{\Gamma_1} q u^* d\Gamma + \int_{\Gamma_2} u q^* d\Gamma + \int_{\Gamma_1} \hat{u} q^* d\Gamma \tag{3}$$

or:

$$u(\xi) + \int_{\Gamma_2} u q^* d\Gamma + \int_{\Gamma_1} \hat{u} q^* d\Gamma = \int_{\Gamma_2} \hat{q} u^* d\Gamma + \int_{\Gamma_1} q u^* d\Gamma, \qquad \xi \in \Omega \tag{4}$$

where $(\xi)$ is a source point.

The term $(u^*)$ is the fundamental solution satisfying Laplace equation that represents a field generated by a singular source at some point $(\xi)$. Hence, at a field point $(x)$, $(u^*)$ must satisfy:

$$\nabla^2 u^* + \delta(x-\xi) = 0 \tag{5}$$

The solution to Eq. (5) for a two-dimensional isotropic domain is:

$$u^* = -\frac{1}{2\pi}\ln(r) \tag{6}$$

$$q^* = -\frac{1}{2\pi r^2}(x-\xi)\cdot n \tag{7}$$

where $r = |x-\xi|$ is the distance between the source point $(\xi)$ and any point of interest $(x)$. Allowing the boundary to be along $(x)$ and rewriting Eq. (4) before the application of boundary conditions:

$$u(\xi) + \int_{\Gamma_x} q^*(x,\xi)u(x)d\Gamma_x = \int_{\Gamma_x} u^*(x,\xi)q(x)d\Gamma_x, \qquad \xi \in \Omega \tag{8}$$

Integrating Eq. (8) such that the source point, $(\xi)$, is included on the circular boundary of radius $(\varepsilon)$, as $\varepsilon \to 0$, results in the left side integral vanishing. For constant elements the right side integral results in $-\frac{1}{2}u(\xi)$. Thus, Eq. (8) can be rewritten as:

$$\frac{1}{2}u(\xi) + \int_{\Gamma_x} q^*(x,\xi)u(x)d\Gamma_x = \int_{\Gamma_x} u^*(x,\xi)q(x)d\Gamma_x, \qquad \xi \in \Omega \tag{9}$$

### 2.2. CONSTANT ELEMENT BOUNDARY DISCRETIZATION

Any boundary $\Gamma$ can be discretized into boundary elements $\Gamma_i$ consisting of nodes at which a value of either $(u)$ or $(q)$ is known and assumed polynomial shape functions between nodes. In this work, only boundary elements with constant shape functions are used.

These elements contain one node per element, leading to the following discretization:

$$u(x) = u_i \Phi(x) \tag{10}$$

$$q(x) = q_i \Phi(x) \tag{11}$$

where $\{u_i\}$ and $\{q_i\}$ are the vectors of nodal values of $(u)$ and $(q)$, respectively, at node $(i)$ and $\Phi(x)$ is the vector of constant shape functions. The discretized Eq. (9) is written as:

$$\frac{1}{2}u_i + \sum_{Elements} u_i \int_{\Gamma_x} q^*(x,\xi)\Phi(x)d\Gamma_x = \sum_{Elements} q_i \int_{\Gamma_x} u^*(x,\xi)\Phi(x)d\Gamma_x \tag{12}$$

Eq. (12) is written in a matrix form:

$$Hu = Gq \tag{13}$$

where matrix $[H]$ satisfies the rigid body motion. Eq. (13) is rearranged and solved as:

$$Ax = f \tag{14}$$

The terms of $[H]$ and $[G]$ matrices can either be determined explicitly or are computed numerically, by numerical integration using Taylor series expansion.

## 3. Taylor Series Expansion

A function can be expressed as a polynomial in terms of its derivatives at some point $(a)$ using Taylor series expansion [Taylor, 1715]:

$$f(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^m(a)}{m!}(x-a)^m \tag{15}$$

where $m \to \infty$.

If the function has a finite amount of nonzero derivatives, it can be integrated exactly:

$$\int_x f(x)dx = \int_x [\frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^n(a)}{n!}(x-a)^n]dx \quad (16)$$

where $(n)$ corresponds to the last nonzero derivative of the function. Since a function $f(x)$ is represented by a polynomial, its integration can be performed:

$$\int_x f(x)dx = \left[ f(a)x + \frac{f'(a)}{2}(x-a)^2 + \frac{f''(a)}{6}(x-a)^3 + ... + \frac{f^n(a)}{(n+1)!}(x-a)^{n+1} \right]_x \quad (17)$$

However, if the function has an infinite amount of nonzero derivatives, integration of the Taylor Series introduces truncation errors, since not all terms in the series can be accounted for.

## 4. Error Analysis on Taylor Series Expansion

A function can also be expressed using Taylor series expansion with remainder as:

$$f(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + R_n \quad (18)$$

where $(n)$ corresponds to the $(n^{th})$ derivative of the function and and $R_n$ is the series remainder as:

$$R_n = \frac{f^n(\zeta)(x-a)^n}{n!} \qquad a < \zeta < x \quad (19)$$

Thus, any function can be integrated exactly as:

$$\int_x f(x)dx = \int_x [\frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + R_n]dx$$
(20)

Hence, truncation error can be defined as:

$$\int_x R_n dx = \int_x f(x)dx - \int_x [\frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1}]dx \tag{21}$$

Integrating Eq. (19) yields:

$$\int_x R_n dx = \left. \frac{f^n(\zeta)(x-a)^{n+1}}{(n+1)!} \right|_x \tag{22}$$

However, the closed form solution of $\int_x R_n dx$ cannot be obtained since $(\zeta)$ is unknown.

The truncation error can be represented by an interval variable. The interval number is a closed set as:

$$\tilde{X} = [\underline{x}, \overline{x}] = \{x \in \Re \mid \underline{x} \le z \le \overline{x}\} \tag{23}$$

The maximum truncation error is found:

$$\max\left\{\int_x R_n dx\right\} = \max\left\{\left. \frac{f^n(\zeta)(x-a)^{n+1}}{(n+1)!} \right|_x\right\} \tag{24}$$

The bounds on the truncation error are computed:

$$E = \max\left\{\left. \frac{f^n(\zeta)(x-a)^{n+1}}{(n+1)!} \right|_x\right\}[-1,1] \tag{25}$$

These interval Taylor series expansion bounds are used in order to represent truncation error of $[H]$ and $[G]$ matrices when numerical integration is not used. The approximate terms of the $[H]$ and $[G]$ matrices for an element of length $(L)$ are computed as:

$$\int_x f(x)dx = \int_x \left[ \sum_{n=1}^{8} \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} \right] dx = \left[ \sum_{n=1}^{8} \frac{f^{n-1}(a)}{(n)!}(x-a)^{n} \right]_0^L \tag{26}$$

## 5. Interval Boundary Element Formulation

The bounds on the exact value of the non-diagonal terms of $[H]$ and $[G]$ matrices are computed using Eqs. (26) and (25). The diagonal terms of the $[H]$ matrix are computed such that the matrix $[H]$ satisfies the rigid body motion constraint. The diagonal terms of the $[G]$ matrix require special consideration since they contain singular integrals, as the distance $r = |x - \xi|$ vanishes at the node. The approximate value of the diagonal terms is computed using Eq. (26).

Since the function is singular at the node, $\max\{f^n(\zeta)\}$ becomes infinite, Eq. (25) cannot be used to meaningfully determine the error bound. The closed form solution of the improper integral of the diagonal terms of the $[G]$ matrix is found, whish is not necessarily in the domain of the actual problem. If the domain of the improper integral is different than that of the problem, the remaining domain is integrated numerically using Eq. (26) and the error found using Eq. (25). If the domain of the improper integral is that of the problem, the difference between the closed form solution and the numerical integration is considered as truncation error.

Interval Boundary Element Analysis using the interval bound on the truncation error is performed as:

$$\tilde{H}\tilde{u} = \tilde{G}\tilde{q} \tag{27}$$

Eq. (14) is rearranged as:

$$\tilde{A}\tilde{x} = \tilde{f} \tag{28}$$

The interval linear system of equation can be solved by Matlab Interval Toolbox [MATLAB 6.5.1], which uses Newton-Krawczyk iteration.

## 6. Discretization error

In the analysis of the discretiztion error, we will look for interval bounded unknown functions that will satisfy the continuous problem.

$$\frac{1}{2}u(\xi)+\int_{\Gamma} q^{*}(x,\xi)u(x)d\Gamma = \int_{\Gamma} u^{*}(x,\xi)q(x)d\Gamma \qquad \xi\in\Gamma \tag{29}$$

The existence and uniqueness of the solution to the above problem for two dimensional Laplace equation when $(u)$ or $(q)$ (but not both) is given is well studied [Friedman 1976]. We will assume that the exact solution to Eq. (29) is $u(x)$ and $q(x)$.

The boundary $\Gamma$ is subdivided into elements. For each element, we will seek the interval values $(\tilde{u})$ and $(\tilde{q})$ that bound the functions $(u)$ and $(q)$ over an element $(i)$ (see Figure 2) such that:

$$\tilde{u}_i \in \left[\underline{u_i},\overline{u_i}\right],\tilde{q}_i \in \left[\underline{q_i},\overline{q_i}\right] \mid \forall\xi \;\; \frac{1}{2}u(\xi)+\sum_i\int_{\Gamma_i} q^{*}(x,\xi)u_i(x)d\Gamma = \sum_i\int_{\Gamma_i} u^{*}(x,\xi)q_i(x)d\Gamma \tag{30}$$

If $(u)$ or $(q)$ are specified as boundary conditions, the bounds of the function are assumed to be given explicitly. Each term of the summation in Eq. (30) is represented graphically in Figure 1.

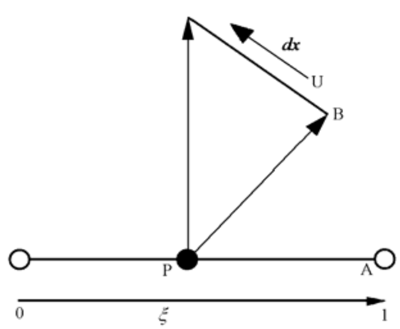



*Figure 1*. Integration from element B from point P on element A.

The integral of the product will be expanded to the product of two intervals: the interval value of u or q and the interval bounds of the integral of the singular solution over the element for all values of $(\xi)$. For example:

$$\int_{\Gamma_i} q^*(x,\xi)u_i(x)d\Gamma \subset \int_{\Gamma_i} q^*(x,\xi)d\Gamma\tilde{u} \qquad (31)$$

if $(q^*)$ has the same sign over the element. If not, the integration domain is subdivided into portions that have the same sign for $(q^*)$. Then the integral is replaced by interval bounds.

$$\int_{\Gamma_i} q^*(x,\xi)d\Gamma u_i \subset \tilde{h}_{ji}\tilde{u} \forall \xi \in \Gamma_j \qquad (32)$$

Eq. (31) is illustrated in Figure 2 schematically.

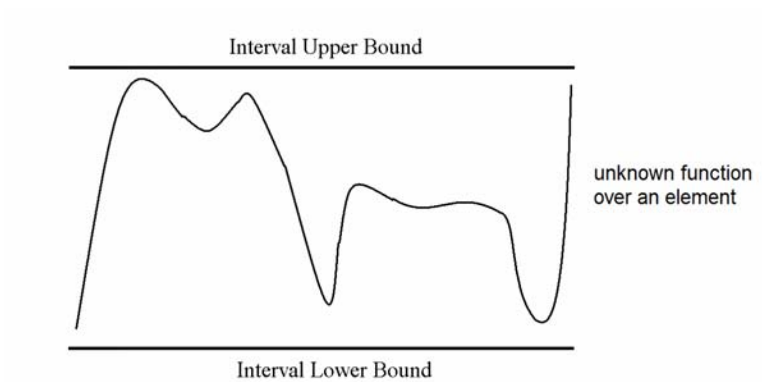



*Figure 2.* Interval bounds on solution to an element.

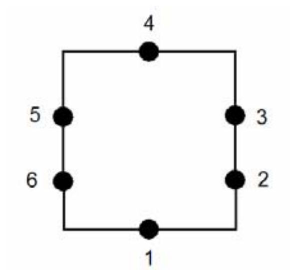
Thus, the interval bounds on the solution of Eq. (30) can be expressed as a generalized interval system of linear equations.

For sharp bounds, the parametric dependence of each row of the $[H]$ or $[G]$ matrices on $(\xi)$ must be included in the solution of the interval system.

## 7. Examples

### 7.1. Example 1

The first example is a demonstration of the interval treatment of uncertain boundary conditions. The unit square domain of the problem as well as the BEA mesh is shown in Figure 3. The left and right hand sides have a zero flux boundary condition while the bottom is between a [0,1] potential and the top is at at a [1,2] potential.




1:1 ratio

*Figure 3.* Boundary discretization using six constant boundary elements.

Boundary Conditions: u1=[0,1], q2=0, q3=0, u4=[1,2], q5=0, q6=0

The interval bounds are shown and compared with the combinatorial solution (Table 1) for the unknown boundary values. In this solution, the interval solution has significantly larger width compared with the combinatorial solution.

We attribute this over estimation to the fact that right hand side in a boundary element solution includes terms that involve products of the interval boundary conditions with terms from the $[H]$ or $[G]$ matrices. Methods for preserving the parameterization of the right hand side vector need to be explored to provide sharper results.

| Node Value | Lower Bound | Combinatorial Lower Bound | Combinatorial Upper Bound | Upper Bound |
|---|---|---|---|---|
| q1 | -2.5770 | -2.0763 | 0.0000 | 0.5007 |
| q2 | 0.0922 | 0.2451 | 1.2451 | 1.3981 |
| u3 | 0.6019 | 0.7549 | 1.7549 | 1.9078 |
| u4 | -0.5007 | 0.0000 | 2.0763 | 2.5770 |
| q5 | 0.6019 | 0.7549 | 1.7549 | 1.9078 |
| q6 | 0.0922 | 0.2451 | 1.2451 | 1.3981 |

*Table 1.* Solutions to Laplace equation with uncertain boundary conditions.

## 7.2. EXAMPLE 2

The second example uses interval BEA to solve Laplace equation on a 2 x 1 domain using six constant boundary elements with a node located at the mid-point (Figure 4). The sides of the domain have zero flux while the bottom is at zero potential and a potential of 50 is at the top. In this example we will use a four point integration method based on a Taylor series to develop interval terms in the $[H]$ and $[G]$ matrices. The interval system of equations is then solved using Matlab.

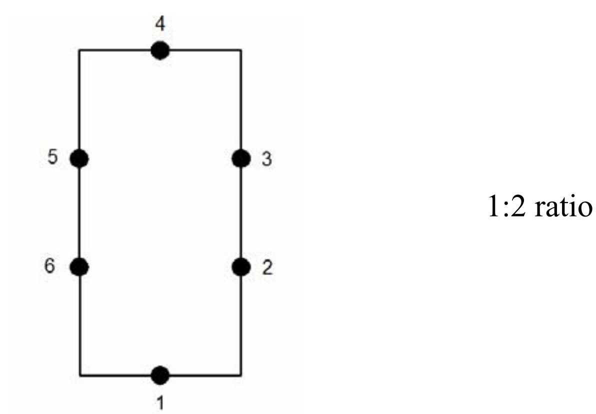



*Figure 4.* Boundary discretization using six constant boundary elements.

Boundary Conditions: u1=0, q2=0, q3=0, u4=50, q5=0, q6=0

The solution obtained by exact integration is shown and compared to the bounds of the solution using the proposed method (Table 2).

| Node Value | Lower Bound | Solution with exact integration | Upper Bound |
|---|---|---|---|
| q1 | -33.6604 | -28.1967 | -23.9615 |
| u2 | 11.1689 | 11.9357 | 12.4285 |
| u3 | 37.5192 | 38.0643 | 38.8833 |
| q4 | 23.4502 | 28.1967 | 34.1717 |
| u5 | 37.5192 | 38.0643 | 38.8833 |
| u6 | 11.1690 | 11.9357 | 12.4285 |

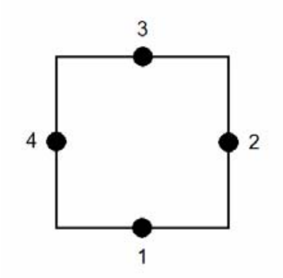
*Table 2.* Solutions to Laplace equation in presence of truncation error.

The results obtained by the present method shows that the presence of truncation errors in integration as well as in solution of the system of linear equations can be bounded using Interval Boundary Element Analysis.

### 7.3. EXAMPLE 3

The third example obtains the bounds on discretization error for the BEA of the Laplace equation. We consider a unit domain with zero flux on each side, a zero potential on the bottom and a unit potential on the top. With the coarse meshes used as well as the need to improve the solution of a parameterized system of interval equations, we will present bounds calculated by a "brut force" construction of interval bounds by constructing terms in the $[H]$ and $[G]$ matrices by moving the point $(\xi)$ over the domain of an element to evaluate terms in Eq. (32). Thus, the results represent the potential to efficiently calculate bounds only of an optimal interval solution method to the parametric problem can be developed.

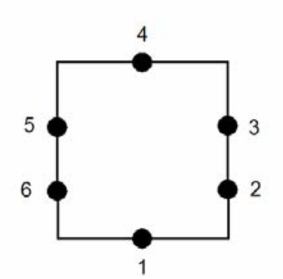
Three different meshes are considered and the solutions in presence of the discretization error are compared.




1:1 ratio

*Figure 5.* Boundary discretization using four constant boundary elements.

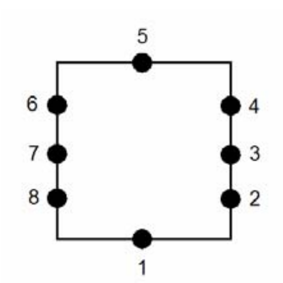
Boundary Conditions: u1=0, q2=0, u3=1, q4=0




1:1 ratio

*Figure 6.* Boundary discretization using six constant boundary elements.

Boundary Conditions: u1=0, q2=0, q3=0, u4=1, q5=0, q6=0




1:1 ratio

*Figure 7.* Boundary discretization using eight constant boundary elements.

Boundary Conditions: u1=0, q2=0, q3=0, q4=0, u5=1, q6=0, q7=0, q8=0

The bounds of the interval BEA solution are shown and compared with a conventional BEA solution where the node of each element is located at its mid-point of the element for the three different meshes (Tables 3-5).

| Node Value | Lower Bound | Middle Value | Upper Bound | Width | Mid-point Node Solution |
|---|---|---|---|---|---|
| q1 | -1.9896 | -1.2512 | -0.5129 | 1.4768 | -1.1746 |
| u2 | 0.0000 | 0.5000 | 1.0000 | 1.0000 | 0.5000 |
| q3 | 0.5129 | 1.2512 | 1.98961 | 1.4768 | 1.1746 |
| u4 | 0.0000 | 0.5000 | 1.0000 | 1.0000 | 0.5000 |

*Table 3.* Solutions to Laplace equation in presence of dicretization error for a four node mesh.

| Node Value | Lower Bound | Central Value | Upper Bound | Width | Mid-point Node Solution |
|---|---|---|---|---|---|
| q1 | -1.4389 | -1.0823 | -0.7258 | 0.7131 | -1.0382 |
| u2 | -0.0793 | 0.2431 | 0.5655 | 0.6448 | 0.2451 |
| u3 | 0.4345 | 0.7569 | 1.0793 | 0.6448 | 0.7549 |
| q4 | 0.7258 | 1.0823 | 1.4389 | 0.7131 | 1.0382 |
| u5 | 0.4345 | 0.7569 | 1.0793 | 0.6448 | 0.7549 |
| u6 | -0.0793 | 0.2431 | 0.5655 | 0.6448 | 0.2451 |

*Table 4.* Solutions to Laplace equation in presence of dicretization error for a six node mesh.

| Node Value | Lower Bound | Central Value | Upper Bound | Width | Mid-point Node Solution |
|---|---|---|---|---|---|
| q1 | -1.2737 | -1.0397 | -0.8057 | 0.4680 | -1.0161 |
| u2 | -0.0731 | 0.1539 | 0.3808 | 0.4539 | 0.1639 |
| u3 | 0.2856 | 0.5000 | 0.7144 | 0.4288 | 0.5000 |
| u4 | 0.6192 | 0.8461 | 1.0731 | 0.4539 | 0.8361 |
| q5 | 0.8057 | 1.0397 | 1.2737 | 0.4680 | 1.0161 |
| u6 | 0.6192 | 0.8461 | 1.0731 | 0.4539 | 0.8361 |
| u7 | 0.2856 | 0.5000 | 0.7144 | 0.4288 | 0.5 |
| u8 | -0.0731 | 0.1539 | 0.3808 | 0.4539 | 0.1639 |

*Table 5.* Solutions to Laplace equation in presence of discretization error for a eight node mesh.

The bounds on the discretization error are fairly sharp and enclose the exact solution for this problem. In fact, for the edges of the 4 element mesh, the bounds are sharp. In addition, the results show that the width of discretization error bounds reduces with mesh refinement.

## 8. Conclusion

In this work, new methods are presented to perform boundary element analysis in the presence of the truncation and discretization errors as well as uncertain boundary conditions. The methods rely on interval methods to quantify local errors in BEA. The examples presented demonstrate the potential of interval based boundary element methods to provide reliable engineering computations. Further work is needed to optimally solve the parametric form of the interval equations to advance interval based BEA to a truly reliable and efficient engineering analysis tool.

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