Outlier Detection in Geodetic Applications with respect to Observation Imprecision

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Abstract. The monitoring of buildings, slide slopes and crustal movements is a central task of geodetic engineering. The aim is the generation of meaningful motion and deformation models in order to quickly and specifically initiate constructional or geotechnical safety measures. The adequateness of the actions depends essentially on the quality of the observation and analysis techniques. Therefore it is important to correctly derive the model parameters and their uncertainty budget considering that the model parameters are typically estimated from a large number of heterogeneous and redundant observations by means of a least-squares adjustment. Here, the uncertainty budget is assumed to comprise both random variability and remaining systematics (imprecision). In practice, there are outliers in the data which have to be detected and eliminated. In conventional techniques only random effects are taken into account. When imprecision is considered additionally, the test strategies have to be extended accordingly. In this study imprecise extensions are obtained for the estimated outliers which are tested statistically using one- and multidimensional hypotheses. The applied procedure is outlined in detail showing both theory and numerical examples.

Keywords: outlier detection, imprecision, geodetic applications, adjustment, hypothesis testing

1. Introduction

In many engineering applications parameters are estimated from a large number of heterogeneous and redundant observations by means of a least-squares adjustment. The quality of the estimated parameters depends essentially on the adequate consideration of all uncertainties in the measurement and analysis process and on the reliability of the observations. In this paper, the uncertainty budget is assumed to comprise both random variability (stochastics) and remaining systematic effects (imprecision).

The outliers in the data occurring in practice have to be detected and then removed. Therefore the accordance of the collected data with the assumptions met in the model must be checked. This requires one- and multidimensional hypotheses tests with imprecise extensions for outlier detection and global tests based on estimated parameters and residuals (see Sections 4 and 5). In this study, the classical test approaches are extended in order to take observation imprecision into account.

The calculation of observation imprecision is based on correction and reduction models applied to the raw observation data. It leads to intervals or fuzzy numbers for their description (see Section 2). The influence of the observation imprecision on the estimated parameters is propagated in a least-squares adjustment (see Section 3). The procedure and the criteria for the test decisions are shown in the context of fuzzy theory. They can be directly applied to pure interval mathematics. The presented approach is transferable to many other engineering applications.

Interval mathematic is an appropriate solution to describe observation imprecision by a real interval $[a]$ consisting of an upper bound a_u and a lower bound a_l or by a centre point a_m and radius a_r. The possibility of variation inside the interval demonstrates the absent knowledge about the correct value, cf. (Schön and Kutterer, 2005b). Intervals can also be defined by a suitable indicator function:

$$i_{[a]}(x) = \begin{cases} 1, & a_l \leq x \leq a_u \\ 0, & \text{else.} \end{cases} \tag{1}$$

Fuzzy-theory was founded by (Zadeh, 1995). It is an extension of the classical set theory. In the classical set theory the membership degree is either 1 (is element) or 0 (is not element). A fuzzy set $\tilde{A}$ is uniquely defined by its membership functions $m_{\tilde{A}}(x)$ over a classical set X (e. g. $X = \mathbb{R}$) with a membership degree between 0 and 1:

$$\tilde{A} := \{(x, m_{\tilde{A}}(x)) \mid x \in X\} \qquad \text{with} \qquad m_{\tilde{A}} : X \to [0,1]. \tag{2}$$

Three basic notions are relevant in the following (see Fig. 1):

$$\begin{aligned} &\text{the } \alpha\text{-cut} && \tilde{A}_\alpha && := \{x \in X \mid m_{\tilde{A}}(x) \geq \alpha\} \text{ with } \alpha \in [0,1], && \text{(3a)} \\ &\text{the support} && supp(\tilde{A}) && := \{x \in X \mid m_{\tilde{A}}(x) > 0\}, && \text{(3b)} \\ &\text{the core} && core(\tilde{A}) && := \tilde{A}_1. && \text{(3c)} \end{aligned}$$

It is obvious that α-cuts are classical sets. In case of convex fuzzy sets (monotonously decreasing reference functions), α-cuts are intervals. The integral over all α-cuts equals the membership function of a fuzzy set:

$$m_{\tilde{A}}(x) = \int_0^1 m_{\tilde{A}_\alpha}(x) d\alpha. \tag{4}$$

In geodetic data analysis, fuzzy numbers and fuzzy intervals are meaningful as they are convex fuzzy sets based on real numbers. Their core is either a single element (fuzzy number) which may refer to a particular observed or derived value or a classical interval which refers to a set of values (fuzzy intervals). In engineering applications LR- and LL-fuzzy numbers and intervals are of particular interest. LR-fuzzy numbers and intervals are defined by their left and right reference functions (see Eq. 5 for a LR-fuzzy interval). LR-fuzzy numbers or intervals with the same left and right reference functions are called LL-fuzzy numbers or intervals.

$$m_{\tilde{A}}(x) = \begin{cases} L\left(\frac{x_m - x - r}{c_l}\right), & x < x_m - r \\ 1, & x_m - r \leq x \leq x_m + r \\ R\left(\frac{x - x_m - r}{c_r}\right), & x > x_m + r \end{cases} \tag{5}$$

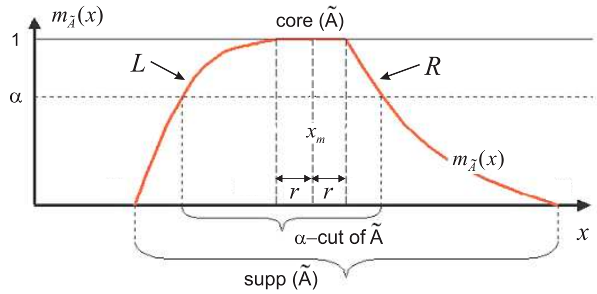


Figure 1. Fuzzy set (LR-fuzzy interval)

with x_m denoting the midpoint of the fuzzy interval, r its radius (see Fig. 1) and c_l, c_r the spread parameters of the reference functions. L-fuzzy numbers are obtained for $L = R$ and $c_l = c_r$. For further information on interval mathematics see (Alefeld and Herzberger, 1983; Jaulin et. al., 2001; Moore, 1979) and on fuzzy-theory, cf. (Bandemer and Näther, 1992; Dubois and Prade, 1980; Viertl, 1996) and (Zadeh, 1995). Studies of fuzzy data analysis in the geodetic context are presented by, e. g., (Kutterer, 2002) and (Schön and Kutterer, 2005b).

2. Observation intervals by means of a sensitivity analysis

Recently, many procedures have been introduced to calculate observation intervals in engineering applications, cf. e. g., (Braems et. al., 2000; Kieffer et. al., 2000; Morales and Son, 1998) and (Muhanna and Mullen, 2001). In geodetic data analysis, observations have to be preprocessed before they can be used for further calculation, e. g., in a least-squares adjustment. For this reason the definition of the observation intervals in geodesy is based on the correction and reduction steps for the raw observations which are based on observation error models. The applied procedure is described in detail in (Schön, 2003). The basic aspects are briefly summarized in the following.

Due to the imperfect knowledge of the influence factors of the preprocessing steps, the reduced observations are afflicted with two types of uncertainties: their stochastic behavior in terms of random variability and several non reducible remaining systematics (observation imprecision). The possible impact of remaining systematic effects is quantified by means of a sensitivity analysis of observation error models. The factual range is assessed based on expert knowledge and empirical studies. This procedure is in full accordance with international recommendations, cf. (ISO, 1995) (GUM). Note that the treatment of systematic errors is different as the GUM proposes variance propagation. An example for distance measurements was shown in (Schön, 2003). The computation in case of GPS measurements was presented in (Schön and Kutterer, 2005b). In this study, four types of observations are of particular interest: distance measurements $\mathbf{l}_{dist}$, direction measurements $\mathbf{l}_{dir}$, zenith angle measurements $\mathbf{l}_z$ and GPS measurements $\mathbf{l}_{GPS}$.

3. Interval calculations for the parameters with a least-squares adjustment

The aim of geodetic applications is the estimation of parameters of interest from the observations, e. g., point coordinates (see Sect. 6 and (Koch, 1999)), deformation fields or strain tensors. Like in many other engineering application, the standard algorithm is a least-squares adjustment using a large number of heterogeneous and redundant observations. The typically non-linear observation equations are linearized in order to use linear model theory. The estimated parameters $\hat{\mathbf{x}}$ are obtained as

$$\mathbf{d}\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{P}\mathbf{A})^+\mathbf{A}^T\mathbf{P}(\mathbf{l} - \mathbf{a_0}), \tag{6a}$$

$$\hat{\mathbf{x}} = \mathbf{x_0} + \mathbf{d}\hat{\mathbf{x}}, \tag{6b}$$

with the $n \times u$ configuration matrix $\mathbf{A}$, the number of unknown parameters u, the number of observations n, the $n \times n$ weight matrix $\mathbf{P}$ (i. e. the inverse of the variance covariance matrix (VCM) of the observations $\mathbf{C_{ll}}$), the $n \times 1$ vector of observations $\mathbf{l}$ and the vector of approximate values $\mathbf{a_0}$. In geodetic networks the normal equation matrix $\mathbf{A}^T\mathbf{P}\mathbf{A}$ can be rank-deficient due to an incomplete definition of the coordinate frame through the configuration. If for example such a network is composed of distance observations only it is not possible to estimate coordinates which are required in practice. The value of rank deficiency is denoted with d. This problem can be overcome when the pseudoinverse matrix $(\mathbf{A}^T\mathbf{P}\mathbf{A})^+$ is used. A standard reference on parameter estimation (and hypotheses tests) is (Koch, 1999).

In case of observation imprecision, we assume the vector of observations as a symmetric interval $[\mathbf{l}]$, with its midpoints $\mathbf{l_m}$ and interval radii $\mathbf{l_r}$, calculated by means of a sensitivity analysis (see Sect. 2). The midpoint of the interval vector $[\mathbf{l}]$ is carrier of the randomness, the remaining systematics of the analysis process are described by the vector of interval radii.

The observation imprecision is propagated to the coordinates by interval extension of the least-squares estimator, cf. (Schön and Kutterer, 2005a),

$$[\mathbf{d}\hat{\mathbf{x}}] = (\mathbf{A}^T\mathbf{P}\mathbf{A})^+\mathbf{A}^T\mathbf{P}([\mathbf{l}] - \mathbf{a_0}), \tag{7a}$$

$$[\hat{\mathbf{x}}] = \mathbf{x_0} + [\mathbf{d}\hat{\mathbf{x}}], \tag{7b}$$

with the assumed precise vector of approximate values and point matrices for $\mathbf{A}$ and $\mathbf{P}$. The vector $\mathbf{a_0}$ can also be chosen as an interval vector in order to take model uncertainties into account. Thus, both observation imprecision and model uncertainties can be treated with interval mathematical methods. Let $\mathbf{y} = \mathbf{l} - \mathbf{a_0}$ the vector of reduced observations ("observed minus computed"), then

$$[\mathbf{y}] = [\mathbf{l}] - \mathbf{a_0} \tag{8}$$

and

$$\mathbf{y_m} = \mathbf{l_m} - \mathbf{a_0}, \tag{9a}$$

$$\mathbf{y_r} = \mathbf{l_r}. \tag{9b}$$

The parameter vector can be split up in a centre $\mathbf{x_m}$ and radius $\mathbf{x_r}$ part:

$$\hat{\mathbf{x}}_\mathbf{m} = \mathbf{x_0} + (\mathbf{A}^T\mathbf{P}\mathbf{A})^+\mathbf{A}^T\mathbf{P}\mathbf{y_m}, \tag{10a}$$

$$\hat{\mathbf{x}}_\mathbf{r} = |\,(\mathbf{A}^T\mathbf{P}\mathbf{A})^+\mathbf{A}^T\mathbf{P}\,|\,\mathbf{y_r}, \tag{10b}$$

where $|\cdot|$ denotes the element by element absolute value of the matrix. Note that the parameter vector is exact component by component, but it overestimates the correct range, which is in general a convex polytope (zonotope), see (Schön and Kutterer, 2005a).

The residuals $\hat{\mathbf{v}}$ are estimated and treated in a similar way. They are obtained as

$$\begin{aligned}\hat{\mathbf{v}} &= \mathbf{A}\mathbf{d}\hat{\mathbf{x}} - \mathbf{y} \\ &= -\mathbf{C}_{\hat{\mathbf{v}}\hat{\mathbf{v}}}\mathbf{P}\mathbf{y},\end{aligned} \tag{11}$$

with

$$\mathbf{C}_{\hat{\mathbf{v}}\hat{\mathbf{v}}} = \mathbf{C}_{\mathbf{ll}} - \mathbf{A}(\mathbf{A}^T\mathbf{P}\mathbf{A})^+\mathbf{A}^T, \tag{12}$$

the VCM of $\hat{\mathbf{v}}$. The interval extension of $\hat{\mathbf{v}}$ in terms of the midpoint $\hat{\mathbf{v}}_{\mathbf{m}}$ and the radius $\hat{\mathbf{v}}_{\mathbf{r}}$ of the residuals reads as (cf. (Kutterer, 2002)):

$$\hat{\mathbf{v}}_{\mathbf{m}} = -\mathbf{C}_{\hat{\mathbf{v}}\hat{\mathbf{v}}}\mathbf{P}\mathbf{y}_{\mathbf{m}}, \tag{13a}$$

$$\hat{\mathbf{v}}_{\mathbf{r}} = |-\mathbf{C}_{\hat{\mathbf{v}}\hat{\mathbf{v}}}\mathbf{P}|\,\mathbf{y}_{\mathbf{r}}. \tag{13b}$$

Then the minimum sum of the squared residuals is derived as

$$\Omega = \hat{\mathbf{v}}^T\mathbf{P}\hat{\mathbf{v}} = \mathbf{y}^T\mathbf{P}\mathbf{C}_{\hat{\mathbf{v}}\hat{\mathbf{v}}}\mathbf{P}\mathbf{y}. \tag{14}$$

4. One-dimensional hypothesis testing for outlier detection

This section presents hypotheses tests for imprecise data in the one-dimensional case. For a more general context and for a more comprehensive field of engineering applications, the test is described in fuzzy-theory. Intervals are special cases of fuzzy sets. Thus, the tests can be directly applied to the examples given in Section 6. The presented test strategy is based on (Römer and Kandel, 1995) and (Viertl, 1996). It is given in detail in (Kutterer, 2004).

4.1. Test strategy and general test decision criterion

First the regions of acceptance ($\tilde{A}$) and rejection ($\tilde{R} = \tilde{A}^C$) have to be described with fuzzy sets. Here, the presentation is restricted to L-fuzzy intervals which are mostly relevant in the application. Hence, the region of acceptance is given as:

$$m_{\tilde{A}}(x) = \begin{cases} L_A\left(\frac{-k-x}{A_s}\right), & x < -k \\ 1, & -k \le x \le k \\ L_A\left(\frac{x-k}{A_s}\right), & x > k \end{cases} \tag{15}$$

with the constants k and $A_s \neq 0$ to control the shape of the region of acceptance.

Consequently, the L-fuzzy test statistic $\tilde{T}$ with midpoint T_m and radius r is introduced:

$$m_{\tilde{T}}(x) = \begin{cases} L_T\left(\frac{T_m - r - x}{T_s}\right), & x < T_m - r \\ 1, & T_m - r \le x \le T_m + r \\ L_T\left(\frac{x - T_m - r}{T_s}\right), & x > T_m + r \end{cases} \tag{16}$$

with $T_s \neq 0$ the spread parameter of the test statistics.

Then the degree of agreement $\gamma_{\tilde{R}}(\tilde{T})$ of the test statistic with the region of rejection and the degree of disagreement $\delta_{\tilde{A}}(\tilde{T}) = 1 - \gamma_{\tilde{A}}(\tilde{T})$ of the test statistics with the region of acceptance are computed. With $F(\mathbb{R})$ the space of fuzzy sets over $\mathbb{R}$ and $F(\mathbb{R} \times \mathbb{R})$ the space of fuzzy sets over $\mathbb{R} \times \mathbb{R}$, the degree of agreement $\gamma : F(\mathbb{R} \times \mathbb{R}) \rightarrow [0, 1]$ of a non empty fuzzy set $\tilde{M} \in F(\mathbb{R})$ with a fuzzy set $\tilde{N} \in F(\mathbb{R})$ ist defined by:

$$\gamma_{\tilde{N}}(\tilde{M}) := \gamma(\tilde{M}, \tilde{N}) = \frac{h(\tilde{M} \cap \tilde{N})}{h(\tilde{M})}. \tag{17}$$

The class of functions $h : F(\mathbb{R}) \rightarrow [0, \infty)$ is defined by the conditions

$$\tilde{U} = \oslash \Leftrightarrow \quad h(\tilde{U}) = 0, \tag{18a}$$

$$\tilde{U} \subseteq \tilde{V} \Leftrightarrow h(\tilde{U}) \le h(\tilde{V}), \tag{18b}$$

with $\oslash$ the empty set. Examples for the class of functions are given in Section 4.2.

Now, the hypotheses for the imprecise test statistics $(\tilde{T})$ have to be introduced. The hypotheses considered here are:

$$\begin{array}{l} H_0 : E(T_m) = \mu = \mu_0 \\ H_A : E(T_m) = \mu = \mu_0 + \delta, \quad \delta \neq 0 \\ \text{with } \; T_m \sim N(\mu, 1) \end{array}$$

The expected value of the midpoint of the test statistics T_m, which describes the stochastic behavior, follows a standardized normal distribution N (under H_0). The presented test strategy also allows to handle empirical test values (e. g. t-distribution) and imprecise variances. The degree of rejectability $\rho_{\tilde{R}}(\tilde{T})$ of the null hypothesis H_0 is then given by

$$\rho_{\tilde{R}}(\tilde{T}) := min(\gamma_{\tilde{R}}(\tilde{T}), \delta_{\tilde{A}}(\tilde{T})). \tag{19}$$

It is compared with a precise critical value ρ_{crit}, what leads to the test decision:

$$\rho_{\tilde{R}}(\tilde{T}) \left\{ \begin{array}{c} \le \\ > \end{array} \right\} \rho_{crit} \in [0, 1] \Longrightarrow \left\{ \begin{array}{c} \text{do not reject } H_0 \\ \text{reject } H_0 \end{array} \right. \tag{20}$$

The imprecise test statistics $\tilde{T}$ is only rejected if it both agrees with $\tilde{R}$ *and* does not agree with $\tilde{A}$.

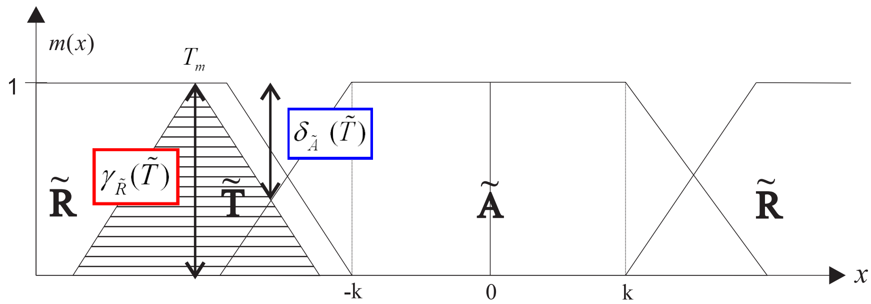


Figure 2. Geometric interpretation of the *height* criterion with a L-fuzzy test value ($r = 0$)

4.2. Two test decision criterions

Now a suitable choice for the class of functions h in Eq. (17) has to be introduced. Section 4.2.1 describes the *height* criterion and Section 4.2.2 the *card* criterion. The *height* criterion allows an easy-to-handle test decision in case of complex fuzzy sets and the *card* criterion allows a better description of the degree of agreement between two fuzzy sets from a practical point of view, cf. (Kutterer, 2004).

4.2.1. *The height criterion*

For the *height* criterion, the function h in Equation (17) is defined as:

$$h(\tilde{U}) = height(\tilde{U}). \tag{21}$$

If the region of acceptance ($\tilde{A}$) and the test statistics ($\tilde{T}$) are L-fuzzy intervals this leads to the degree of rejectability:

$$\rho_{\tilde{R}}(\tilde{T}) = \min\,(\gamma_{\tilde{R}}(\tilde{T}), \delta_{\tilde{A}}(\tilde{T})) = \begin{cases} 0, & core(\tilde{T}) \cap core(\tilde{A}) \neq \oslash \\ \delta_{\tilde{A}}(\tilde{T}), & core(\tilde{T}) \cap core(\tilde{A}) = \oslash \end{cases} \tag{22}$$

with $\delta_{\tilde{A}}(\tilde{T}) = 1\text{-}height(\tilde{T} \cap \tilde{A})$ and $\gamma_{\tilde{R}}(\tilde{T}) = height(\tilde{T} \cap \tilde{R})$. The geometric interpretation of the test is given in Figure 2. In the case of $core(\tilde{T}) \cap core(\tilde{A}) \neq 0$, the null hypothesis H_0 cannot be rejected ($\rho_{\tilde{R}}(\tilde{T}) = 0$). Here, an exemplary test scenario with $T_m > 0$ is considered.

In case of different types of reference functions for the test statistics and the region of acceptance, the degree of rejectability under H_0 can not be given explicitly, it has to be computed numerically. Therefore the point of intersection x_{num} between the reference function of the test statistics and the reference function for the region of acceptance is computed by root-finding, e. g. using a Newton- or bisection algorithm (see (Jaulin et. al., 2001)).

$$L_T\left(\frac{T_m - r - x}{T_s}\right) - L_A\left(\frac{x-k}{A_s}\right) = 0 \quad \text{for} \quad k \leq x \leq T_m - r. \tag{23}$$

This numerical solution $x_{num} > 0$ can be used for the computation of the degree of rejectability of the null hypothesis H_0:

$$\rho_{\tilde{R}}(\tilde{T}) = \begin{cases} 1, & (\tilde{T} \cap \tilde{A}) = \oslash \\ 1 - L_T\left(\frac{T_m - r - x_{num}}{T_s}\right), & (\tilde{T} \cap \tilde{A}) \neq \oslash \end{cases} \tag{24}$$

Note that the numerical solution x_{num} is a function of the region of acceptance (A_s and k). The test decision in case of $\tilde{T} \cap \tilde{A} \neq \oslash$ is now based on the comparison of the degree of rejectability with the critical value ρ_{crit}, see Eq. (25). In any case the null hypothesis is rejected for $\tilde{T} \cap \tilde{A} = \oslash$. If $(\tilde{T} \cap \tilde{A}) \neq \oslash$,

$$\rho_{\tilde{R}}(\tilde{T}) = 1 - L_T\left(\frac{T_m - r - x_{num}}{T_s}\right) > \rho_{crit} \Longrightarrow \text{ reject } H_0. \tag{25}$$

If the reference functions for the region of acceptance and the test statistics are of same type, the degree of rejectability of the null hypothesis H_0 can be computed explicitly:

$$\rho_{\tilde{R}}(\tilde{T}) = \begin{cases} 1, & (\tilde{T} \cap \tilde{A}) = \oslash \\ 1 - L_T\left(\frac{T_m - k - r}{T_s + A_s}\right), & (\tilde{T} \cap \tilde{A}) \neq \oslash \end{cases} \tag{26}$$

In case of $(\tilde{T} \cap \tilde{A}) \neq \oslash$, the test decision is now described by:

$$\rho_{\tilde{R}}(\tilde{T}) = 1 - L_T\left(\frac{T_m - k - r}{T_s + A_s}\right) > \rho_{crit} \Longrightarrow \text{ reject } H_0. \tag{27}$$

4.2.2. *The card criterion*

The *card* criterion is a second possibility for the function h in Equation (17):

$$h(\tilde{U}) = card(\tilde{U}) := \int_{\mathbb{R}} m_{\tilde{U}}(x)dx. \tag{28}$$

The *card* criterion gives a suitable description of the agreement between two fuzzy sets, but the computational complexity is much higher then using the *height* criterion, in particular for complex fuzzy sets. The degree of rejectability of H_0 has to be computed based on the cardinality of the fuzzy sets resulting from the intersection of the test statistics and the region of acceptance ($card(\tilde{T} \cap \tilde{A})$) and region of rejection ($card(\tilde{T} \cap \tilde{R})$), respectively; see Eq. (29). Figure 3 shows a geometric interpretation of the *card* criterion.

$$\begin{gathered} \rho_{\tilde{R}}(\tilde{T}) := min(\gamma_{\tilde{R}}(\tilde{T}), \delta_{\tilde{A}}(\tilde{T})), \\ \text{with } \gamma_{\tilde{R}}(\tilde{T}) = \frac{card(\tilde{T} \cap \tilde{R})}{card(\tilde{T})} \text{ and } \delta_{\tilde{A}}(\tilde{T}) = 1 - \frac{card(\tilde{T} \cap \tilde{A})}{card(\tilde{T})}. \end{gathered} \tag{29}$$

In case of classical intervals for the regions of acceptance, the degree of rejectability of the null hypothesis H_0 is now easy to handle and reads:

$$\rho_{\tilde{R}}(\tilde{T}) = \gamma_{\tilde{R}}(\tilde{T}) = \delta_{\tilde{A}}(\tilde{T}) = \begin{cases} 1, & (\tilde{T} \cap \tilde{A}) = \oslash \\ 1 - \frac{card(\tilde{T} \cap \tilde{A})}{card(\tilde{T})}), & (\tilde{T} \cap \tilde{A}) \neq \oslash. \end{cases} \tag{30}$$

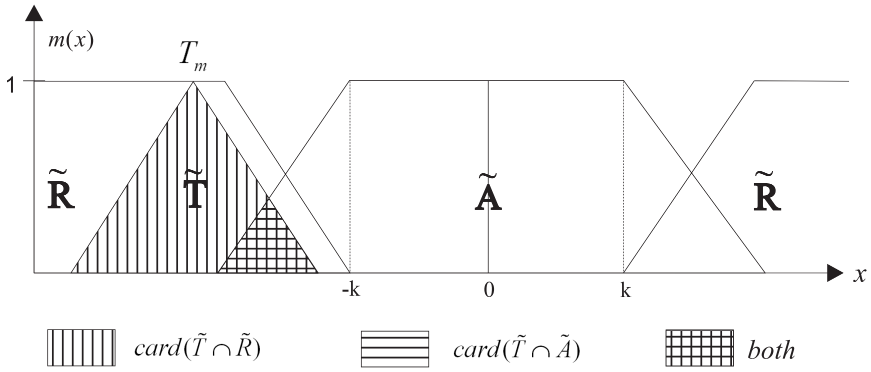


Figure 3. Geometric interpretation of the *card* criterion with a L-fuzzy test value ($r = 0$)

5. Global and multiple tests using α-cut optimization

5.1. The pure stochastic case

The pure stochastic case in multidimensional hypothesis is well known in many engineering applications, cf. (Koch, 1999). The test is based on a quadratic form $\mathbf{z}^T\mathbf{M}\mathbf{z}$ with $\mathbf{z}$ a n×1 vector and $\mathbf{M}$ a semi-positive definite symmetric matrix. With the expected value $E(\mathbf{z}) = \mu$ and its VCM $\mathbf{C_{zz}}$, the expected value of the quadratic form is given by:

$$E(\mathbf{z}^T\mathbf{M}\mathbf{z}) = trace(\mathbf{M}\mathbf{C_{zz}}) + \mu^T\mathbf{M}\mu. \tag{31}$$

If $\mathbf{M}\mathbf{C_{zz}}$ is idempotent and $\mathbf{z}$ is normal distributed according to $\mathbf{z} \sim N(\mu, \mathbf{C_{zz}})$, the quadratic form $\mathbf{z}^T\mathbf{M}\mathbf{z}$ follows a non-central χ^2-distribution, cf. (Koch, 1999):

$$\mathbf{z}^T\mathbf{M}\mathbf{z} \sim \chi^2(rank(\mathbf{M}), \mu^T\mathbf{M}\mu) = \chi^2(f, \lambda), \tag{32}$$

with $f = rank(\mathbf{M})$ the degrees of freedom and $\lambda = \mu^T\mathbf{M}\mu$ the non-centrality parameter.

From the results of a least-squares adjustment, the quadratic form may be given by the Equation (14) that follows a central $\chi^2(n-u+d, 0)$-distribution ($\lambda = 0$) with $n-u+d$ degrees of freedom:

$$\mathbf{y}^T(\mathbf{P}\mathbf{C}_{\hat{\mathbf{v}}\hat{\mathbf{v}}}\mathbf{P})\mathbf{y} \sim \chi^2(f, 0) \quad \text{with} \quad f = n-u+d \quad \text{under the null hypothesis} \quad H_0 : E(\hat{\mathbf{v}}) = \mathbf{0}. \tag{33}$$

5.2. Global and multiple tests with observation imprecision

Now Eq. (14) has to be treated with fuzzy techniques with a given imprecise vector of reduced observations $\tilde{\mathbf{y}}$, e. g. from Section 2. We consider intentionally point matrices for $\mathbf{C}_{\hat{\mathbf{v}}\hat{\mathbf{v}}}$ and $\mathbf{P}$. Each kind of model uncertainty is transformed into the imprecise vector of observations, cf. (Schön and Kutterer, 2005a). The fuzzy evaluation of the quadratic form

$$\Omega = \mathbf{y}^T(\mathbf{P}\mathbf{C}_{\hat{\mathbf{v}}\hat{\mathbf{v}}}\mathbf{P})\mathbf{y} \tag{34}$$

is based on Zadeh's extension principle. If the quadratic form fulfills the criteria

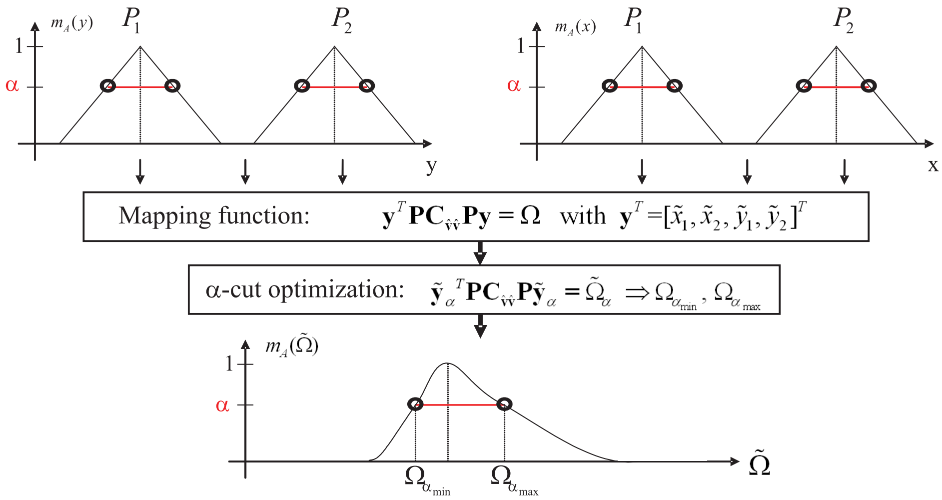


Figure 4. α-cut optimization for a point test

- convexity of the quadratic function (semi-positive definite matrix $\mathbf{PC}_{\hat{\mathbf{v}}\hat{\mathbf{v}}}\mathbf{P}$)
- continuity onto mapping (e. g. no change of the algebraic sign)
- convex input fuzzy sets

the extension principle can be replaced by a min-max operator of an optimization problem, cf. (Möller and Beer, 2004). The properties above are given in a least-squares adjustment with convex fuzzy numbers or fuzzy vectors and play a key role for a strict realization of the extension principle with an optimization problem. Furthermore, in case of a convex function local optimization problems can be applied. We propose a recursive Newton algorithm for minimizing/maximizing a quadratic function subject to bounds of the variables, cf. (Coleman and Li, 1996). In case of observation intervals, the optimization algorithm has to be applied only once. For fuzzy input variables the optimization algorithm is applied for a sufficient number of α-cuts of the input variables to compute the min-max values for the associated α-cut of the fuzzy output variable. The minimum and maximum values of each α-cut are given by $\Omega_{\alpha_{min}}$ and $\Omega_{\alpha_{max}}$ and the test statistics is constructed as $\tilde{T} = \tilde{\Omega}$. Figure 4 shows an example of α-cut optimization for a point test in the two-dimensional space.

Now the test strategy from Section 4 is applied, what leads for the *card*-criterion to the test scenario given in Figure 5. The test hypotheses and the test decision, respectively, are given by:

$$H_0 : E(\hat{\mathbf{v}}_\mathbf{m}) = \mathbf{0}$$
$$H_A : E(\hat{\mathbf{v}}_\mathbf{m}) \neq \mathbf{0}$$
$$\text{with } \; T_m \sim \chi^2(f, 0) \; \text{ and } \; f = n - u + d$$

$$\rho_{\tilde{R}}(\tilde{T}) = \left\{ \begin{array}{c} \leq \\ > \end{array} \right\} \rho_{crit} \in [0,1] \Longrightarrow \left\{ \begin{array}{c} \text{do not reject } H_0 \\ \text{reject } H_0 \end{array} \right. \tag{35}$$

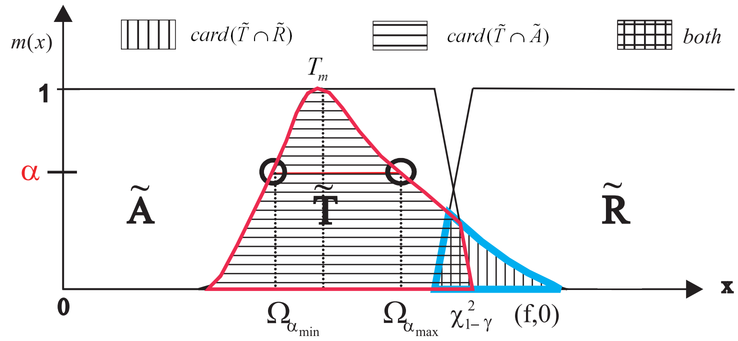


Figure 5. χ^2-test with the *card* criterion

In the case of $\rho_{\tilde{R}}(\tilde{T}) > \rho_{crit}$, the null hypothesis H_0 is rejected and one can assume, that there are outliers in the observations. Further reasons for the rejection of the null hypothesis are non suitable choices of the functional or stochastic modell-components. Therefore each observation or multiple observations has to be tested using one- and multidimensional tests to detect the outliers in the data.

Note, it is also possible to refer this problem directly to the influence factors of a sensitivity analysis, even though it has not been shown formally.

6. Examples with geodetic applications

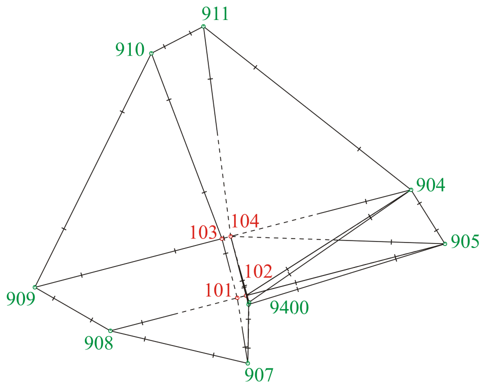
Now selected examples for outlier detection in a three dimensional geodetic network for the monitoring of the lock Uelzen I are shown. We focus our presentation on the multidimensional case because the one-dimensional is straightforward from the given test specifications. Due to the imprecise vector of observations (see Sect. 2), the *card* criterion is used for the test decisions. The regions of acceptance are given by classical intervals with a significance level of $\gamma = 5\%$. The critical value ρ_{crit} is chosen as 0.5. Note that all numerical examples for the test statistics presented in this section are based on the *support* of the test statistics ($supp(\tilde{T})$) in order to have a clearer representation.

Figure 6.a shows the lock and Figure 6.b the geometric configuration of the geodetic monitoring network. The network is composed of eight control points around the lock and four object points (101-104) on top of the lock. The aim is the formulation of a meaningful deformation model for the object points in order to quickly and specifically initiate constructional or geotechnical safety measures. Therefore different measurements between the network points are carried out with special geodetic equipment such as GPS receivers and automatic tacheometers. Typical geodetic examples for the collected measurements are horizontal directions, zenith angles, distances and GPS baselines.

Table I. Interval radii and standard deviations of the observations

	Distances	Directions	Zenith angles	GPS baselines
l_r	2.0 mm	0.3 mgon	0.5 mgon	0.2 mm
σ	5.0 mm	1.0 mgon	3.0 mgon	3.0 mm

a. The lock Uelzen I



b. The geodetic monitoring network

Figure 6. Lock Uelzen

After the computation of the observation intervals based on the uncertainty of the measurements and the preprocessing steps (cf. Sect. 2), the uncertainty budget is transferred to the parameters of interest (cf. Sect. 3). Here, the parameters of interest are the 3-d point coordinates of the geodetic network points which are estimated in a least-squares adjustment. The orders of magnitude of the interval radii and the standard deviations of the observations are given in table I (for the presented examples).

6.1. Problem definition

Outliers in the collected measurements may falsify point coordinates. Consequently they don't show the actual movements of the lock points. This may prevent a proper initiation of constructional or geotechnical safety measures. For this reason, the outliers in the data have to be detected and then removed. A general strategy which is typically used in Geodesy were presented by (Baarda, 1968). This strategy uses standardized residuals for the test decision in the one-dimensional case (data snooping):

$$T = \frac{\hat{v}_i}{\sigma_{\hat{v}_i}} \sim N(0,1) \quad \text{with} \quad H_0 : E(\hat{v}_i) = 0, \quad H_A : E(\hat{v}_i) \neq 0 \tag{36}$$

with the estimated residual $\hat{v}_i$, its standard deviation $\sigma_{\hat{v}_i}$ and the standardized normal distribution $N(0,1)$. In the multidimensional case the given vector of observations is tested within a quadratic form (cf. Section 5.1 and (Koch, 1999)). The test statistics is then given by:

$$T = \hat{\mathbf{v}}^T \mathbf{P} \hat{\mathbf{v}} \sim \chi^2(f, 0) \quad \text{with} \;\; H_0: \; E(\hat{\mathbf{v}}) = \mathbf{0}, \;\; H_A: \; E(\hat{\mathbf{v}}) \neq \mathbf{0}, \tag{37}$$

with $f = n - u + d$ the degree of freedom.
If the value of the test statistics T exceeds the chosen fractile value, the null hypothesis H_0 is rejected and the outlier is considered as revealed. This strategy is standard in geodesy.

6.2. One-dimensional case (distance)

The first example is a one-dimensional test for the observed distance between the control point 910 and the object point 103. The midpoint $\hat{\mathbf{v}}_\mathbf{m}$ and the radius $\hat{\mathbf{v}}_\mathbf{r}$ of the residuals are computed according to the Eq. (13a) and (13b). Each observation i is tested individually and the midpoint T_{m_i} and the radius r_i of the test statistics in the imprecise case read as:

$$T_{m_i} = \frac{\hat{v}_{m_i}}{\sqrt{C_{\hat{v}\hat{v}_{ii}}}} \quad \text{under} \;\; H_0: \; E(\; \hat{v}_{m_i} \;) = 0, \;\; H_A: \; E(\; \hat{v}_{m_i} \;) \neq 0 \tag{38}$$

$$r_i = \frac{\hat{v}_{r_i}}{\sqrt{C_{\hat{v}\hat{v}_{ii}}}} \quad \Rightarrow \quad supp(\tilde{T}_i) = [T_{m_i} - r_i, T_{m_i} + r_i] \tag{39}$$

In this case, the numerical values for the observed distance between the points 910 and 103 are obtained by

$$T_{m_{910-103}} = \frac{\hat{v}_{m_{910-103}}}{\sqrt{C_{\hat{v}\hat{v}_{910-103}}}} = \frac{0.0101\text{m}}{0.0047\text{m}} = 2.131 \tag{40}$$

$$r_{910-103} = \frac{\hat{v}_{r_{910-103}}}{\sqrt{C_{\hat{v}\hat{v}_{910-103}}}} = \frac{0.0041\text{m}}{0.0047\text{m}} = 0.865 \quad \Rightarrow \quad supp(\tilde{T}_{910-103}) = [1.266, 2.996]. \tag{41}$$

Now, the test decision based on the $z_{1-\frac{\gamma}{2}}$ fractile value for the two-sided hypothesis test with $\gamma = 5\%$ reads:

$$\rho_{\tilde{R}}(\tilde{T}) = 0.60 > \rho_{crit} = 0.5 \Longrightarrow \text{reject H}_0. \tag{42}$$

Obviously in case of $\rho_{crit} = 0.5$ the test is rejected, if the midpoint of the symmetric test statistics is outside the region of acceptance $T_m > z_{1-\frac{\gamma}{2}}$. In case of $\rho_{crit} > 0.5$ the midpoint of the test statistics may be outside without rejecting the test, this is caused by taking observation imprecision into account. In case of classical regions of acceptance, the value ρ_{crit} must not be chosen too small because observation imprecision is an additive term of uncertainty.

6.3. Multiple tests (GPS baseline)

Second, a multiple test for a GPS baseline between the points 907 and 908 is presented. According to the pure stochastic case (see (Koch, 1999)), the imprecise quadratic form for the test reads as:

$$\Omega = \mathbf{y}^T (\mathbf{P} \mathbf{C}_{\hat{\mathbf{v}}\hat{\mathbf{v}}} \mathbf{P}) (\mathbf{B} (\mathbf{B}^T \mathbf{P} \mathbf{B})^{-\mathbf{1}} \mathbf{B}^T) (\mathbf{P} \mathbf{C}_{\hat{\mathbf{v}}\hat{\mathbf{v}}} \mathbf{P}) \mathbf{y} \quad \text{with} \quad \mathbf{y} \in \tilde{\mathbf{y}} \tag{43}$$

and

$$\mathbf{B}^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \quad (44)$$
$$\Delta x_{GPS} \;\; \Delta y_{GPS} \;\; \Delta z_{GPS}$$

The relevant VCM of the observation to be tested in a multiple hypothesis has to be selected from the data. For this reason, the matrix $\mathbf{B}$ is introduced, which is in case of GPS baselines defined by Eq. (44). Hence, the asymmetric imprecise test statistics $\tilde{T}$ can be computed by means of the optimization algorithm (cf. Section 5.2 and (Coleman and Li, 1996)):

$$supp(\tilde{T}) = supp(\tilde{\Omega}) = [9.907, 10.291] \quad \text{with } T_m = 10.097 \sim \chi^2(p, 0) \quad (45)$$
$$H_0: \; E(\hat{\mathbf{v}}_{\mathbf{m}_{GPS}}) = \mathbf{0} \;\text{ and }\; H_A: \; E(\hat{\mathbf{v}}_{\mathbf{m}_{GPS}}) \neq \mathbf{0}$$

The test decision with the fractile value $\chi^2_{p,1-\gamma} = 7.814$ (with $p = 3$ the number of simultaneously tested observations) reads as:

$$\rho_{\tilde{R}}(\tilde{T}) = 1.0 > \rho_{crit} = 0.5 \Longrightarrow \text{reject H}_0 \quad (46)$$

The GPS baseline between the points 907 and 908 is revealed as an outlier and removed from the data. In case of GPS observations in small geodetic networks (< 5km) with less changes in altitude, the observation imprecision is small. For this reason the spreads of the test statistics are tight and close to symmetric.

6.4. Global test in least squares adjustments

In the last example we compute the imprecise global test in least-squares adjustment. The starting procedure is the fuzzy evaluation of Eq. (34) with the described optimization method. The imprecise test statistics $\tilde{T}$ is then given by

$$supp(\tilde{T}) = supp(\tilde{\Omega}) = [306.756, 315.851] \quad \text{with } \tilde{T}_m = 310.211 \sim \chi^2(f, 0) \quad (47)$$
$$H_0: \; E(\hat{\mathbf{v}}_{\mathbf{m}}) = \mathbf{0} \;\text{ and }\; H_A: \; E(\hat{\mathbf{v}}_{\mathbf{m}}) \neq \mathbf{0}$$

and the fractile value for the test decision reads as ($\gamma = 5\%$):

$$\chi^2_{f,1-\gamma} = 310.396 \qquad (f = 271) \quad (48)$$

Hence, the test decision

$$\rho_{\tilde{R}}(\tilde{T}) = 0.579 > \rho_{crit} = 0.5 \Longrightarrow \text{ reject H}_0, \quad (49)$$

shows, that with the given significance level of $\gamma = 5\%$ the global test is rejected, although the midpoint of the specified test statistics is inside of the region of acceptance ($\rho_{crit} = 0.5$). The test rejection is caused by the assymmetric imprecise test statistics which considers the quadratic impact of the imprecise influence parameters on the specified test statistic.

7. Conclusions

In this study, one and multidimensional hypotheses tests in case of observation imprecision are developed. The consideration of observation imprecision is an independent extension of the classical test approach. New approaches for outlier detection are shown, based on the intervals or fuzzy numbers of the observations. The presented test strategy allows to handle with all types of uncertainty, given as imprecise vectors of observations and can be applied to least-squares adjustments in many engineering applications. Thus, it is an essential observation-based contribution to the quality management in engineering.

Furthermore, this paper shows that an automated joint treatment of stochasticity and imprecision from the original observation up to the target parameters is possible. It turns out that remaining systematics have to be taken into account in geodetic data analysis. This allows an improved interpretation of the parameters of interest.

Finally, the presented test strategy allows a numerical calculation of the fractile value $z_{1-\frac{\alpha_{impr}}{2}}$ of the standard normal distribution. The evaluation of type I and type II errors in the imprecise case is possible.

The main focus of the following studies lies on the analysis and reanalysis of simulated and real data sets in order to make more improved decisions, in e.g. about the critical value ρ_{crit}. In addition, more extensive works in numerical computations with the *card* criterion and in the comparison between the *height* and *card* criterion has to be done.

Acknowledgements

The presented paper shows results and new ideas developed during the research project KU 1250/4-1 "Geodätische Deformationsanalysen unter Verwendung von Beobachtungsimpräzision und Objektunschärfe", which is funded by the German Research Foundation (DFG). This is gratefully acknowledged. The third author stays as a Feodor-Lynen-Fellow with F. K. Brunner. He thanks his host for giving the possibility to contribute to this paper and the Alexander von Humboldt Foundation for the financial support.

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