Reliability of Structural Reliability Estimation

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Abstract: In this paper we study the reliability of calculations of the structural reliability. It compares the exact reliability expression within the Bernoulli-Euler column theory with its counterpart obtained via the finite difference expression in the buckling context.

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1. Introduction

The recent decade is characterized in intensive increase of application of probabilistic methods in engineering (Elishakoff *et al*, 2001; Arbocz *at al*, 1995, Chryssanthoupoulos,1998). The matter of the accuracy of the probabilistic design of structures, therefore, becomes of paramount importance. In probabilistic design the main quantity of interest is the structural reliability. Since its calculation involves the numerical calculation the natural question arises on the reliability of the reliability calculation. The paper by Elishakoff (1999) was apparently the first one to address this issue in the structural analysis context. Here we extend Ref. Elishakoff (2001) for the buckling of structures. In particular, we deal with the reliability of finite difference method's application to structural reliability evaluation.

There are several studies that deal with the finite difference evaluation of the buckling phenomenon in deterministic setting. Namely, the papers by Falk (1956), Salvadori (1949), Wifi *et al* (1989) ought be mentioned. Seide (1975) was able, in his seminal paper, to evaluate the analytical expression for the buckling load, when the column is subdivided by N segments. The Seide's formula is a central one in this investigation to study the reliability of the reliability evaluation.

2. Recapitulation of Seide's Solution

The differential equation that governs the buckling of a column of uniform stiffness subjected at the end by a compressive load *P*, reads:

$$EI\frac{d^4w}{dx^4}+P\frac{d^2w}{dx^2}=0 \tag{1}$$

where *EI* is the bending stiffness of the column, *x* is the axial coordinate, *w* is the transversal displacement and *P* is the axial load.

To solve complicated problems, the ordinary differential equations are usually replaced by a set of equivalent algebraic equations that are easier to solve than the differential one. One of such methods, known as finite-difference technique, is based on the fact that a derivative of a function at a point can be approximated by an algebraic expression consisting of the value of the function at that point and at several nearby points. Here, to study the reliability of reliability calculations, we investigate the case of an uniform column that possesses the exact solution, so a direct comparison is possible with the exact solution. We first recapitulate the solution derived by Seide (1995).

In particular, using first order central difference method, under the condition of uniform nodal points spacing and for any nodal point *i* the Eq(1) takes the following expression

$$w_{i-2}-\left(4-\frac{Ph^2}{EI}\right)w_{i-1}+\left(6-2\frac{Ph^2}{EI}\right)w_i-\left(4-\frac{Ph^2}{EI}\right)w_{i+1}+w_{i+2}=0 \tag{2}$$

where *h* is the length of each segment given by the ratio between the length of the bar *L* and the total number *N* of segments.

To solve the difference equation (2) with constant coefficients we can express the solution in the following form:

$$w_i = \mathrm{A}\lambda^i \tag{3}$$

The introduction of the expression (3) into Eq. (2) leads to the resulting equation in λ:

$$\left(\frac{1}{\lambda}+\lambda\right)^2-\left(4-\frac{Ph^2}{EI}\right)\left(\frac{1}{\lambda}+\lambda\right)+4-2\frac{Ph^2}{EI}=0 \tag{4}$$

Eq.4 has the following solutions:

$$\lambda_{1,2} = 1$$
$$\lambda_{3,4} = 1 - \frac{Ph^2}{2EI} \pm \sqrt{\left(\frac{Ph^2}{2EI}\right)^2 - \frac{Ph^2}{EI}} \tag{5}$$

The consideration that

$$\left(\frac{Ph^2}{2EI}\right)^2 - \frac{Ph^2}{EI} = -\left[1 - \left(1 - \frac{Ph^2}{2EI}\right)^2\right] \tag{6}$$

allows to rewrite the solutions $\lambda_{3,4}$ in Eq.(5) in a different way:

$$\lambda_{3,4} = 1 - \frac{Ph^2}{2EI} \pm i\sqrt{1 - \left(1 - \frac{Ph^2}{2EI}\right)^2} \tag{7}$$

Thus the general solution for w_i takes the form:

$$w_i = A_1 + A_2 i + A_3 \cos i\vartheta + A_4 \sin i\vartheta \tag{8}$$

in which A_1, A_2, A_3 and A_4 are arbitrary constants of integration and

$$\vartheta = \cos^{-1}\left(1 - \frac{Ph^2}{2EI}\right) \tag{9}$$

Determination of the four constants of integration is obtained using the four boundary conditions, two at each end of the column. For a simply supported column at both edges we have:

$$w_0 = w_N = 0\ ; \quad w_{-1} = -w_1\ ; \quad w_{N+1} = -w_{N-1} \tag{10}$$

For a clamped columns at both ends the boundary conditions become:

$$w_0 = w_N = 0\ ; \quad w_{-1} = w_1\ ; \quad w_{N+1} = w_{N-1} \tag{11}$$

Let us concentrate on the case of a simply supported column. Introduction of the expression of displacement given by Eq.(8) into the boundary conditions (10) yields:

$$\begin{aligned} &A_1 + A_3 = 0 \\ &A_1 + A_3 \cos\vartheta = 0 \\ &A_1 + A_2 N + A_3 \cos N\vartheta + A_4 \sin N\vartheta = 0 \\ &A_1 + A_2 N + A_3 \cos\vartheta \cos N\vartheta + A_4 \cos\vartheta \sin N\vartheta = 0 \end{aligned} \tag{12}$$

Since the system of equations (12) is homogeneous, the determinant of the coefficients of A_1, A_2, A_3 and A_4 must vanish. The condition to satisfy is the following:

$$4N\left(\sin\frac{\vartheta}{2}\right)^4 \sin N\vartheta = 0 \tag{13}$$

Eq.(13) implies:

$$\sin N\vartheta = 0 \tag{14}$$

which has the solutions

$$N\vartheta = k\pi \qquad k=1,2,3,\ldots \tag{15}$$

From Eq.(9) we can evaluate the expression for $\cos\vartheta$ as follows:

$$\cos\vartheta = \cos\frac{k\pi}{N} = 1 - \frac{Ph^2}{2EI} \tag{16}$$

Using of trigonometric relations we obtain

$$\frac{Ph^2}{EI} = 4\sin^2\frac{k\pi}{2N} \tag{17}$$

in which k should be set equal to unity for the smallest critical load. Keeping in mind that h is the length of each of the N segments and it is the ratio between the length L of the column and the total numbers of segments N, the critical load for a simply supported column at both ends is expressed as follows:

$$P_{cr} = \frac{EI\pi^2}{L^2}\left(\frac{\sin \pi/2N}{\pi/2N}\right)^2 \tag{18}$$

When N tends to infinity we obtain the well-known solution for critical load of a simply supported column. The expression (18) belongs to Seide (1975).

3. Probabilistic Analysis of Seide's Result

Next step is to consider the case in which the elastic modulus of the column can be treated as a continuous random variable with probability distribution function $F_E(e)$ with $e > 0$, assuming that the other parameters are deterministic quantities.

The conventional requirement to avoid buckling phenomenon is that the critical load must be greater or equal than a fixed allowable value P_0

$$P_{cr} \geq P_0 \tag{19}$$

From the expression of P_{cr} given in Eq.(18) we see that if the modulus of elasticity E is a random variable, the left hand side P_{cr} of Eq (19) also becomes a random variable. We are interested in the interval of possible values of E for which the Eq (19) is satisfied. From its definition the reliability R is the probability of the event specified in Eq (19):

$$R = Prob\left(P_{cr} \geq P_0\right) \tag{20}$$

Introducing the expression of P_{cr} given in (18), the Eq.(20) can be rewritten as follows:

$$R = Prob\left[\frac{\pi^2 EI}{L^2}\left(\frac{\sin \pi/2N}{\pi/2N}\right)^2 \geq P_0\right] \tag{21}$$

or

$$R = 1 - Prob\left[\frac{\pi^2 EI}{L^2}\left(\frac{\sin \pi/2N}{\pi/2N}\right)^2 \leq P_0\right] \tag{22}$$

Thus

$$R = 1 - Prob\left[E \le \frac{L^2 P_0}{\pi^2 I} \left(\frac{\pi/2N}{\sin \pi/2N} \right)^2 \right] \tag{23}$$

Given the probability density function of the random variable *E,* Eq.(23) becomes:

$$R = 1 - F_E\left[\frac{L^2 P_0}{\pi^2 I} \left(\frac{\pi/2N}{\sin \pi/2N} \right)^2 \right] \tag{24}$$

where the reliability of the column equals one minus the probability distribution function F_E of the modulus of elasticity at the level $\left(L^2 P_0/\pi^2 I\right)\left[(\pi/2N)/(\sin \pi/2N)\right]^2$.

4. Probabilistic Design of the Column

Once we know the expression of *R* we can pose the design problem of the column, under the consideration that the structure performs acceptably if the reliability exceeds or equals a codified reliability value r_0 :

$$R \ge r_0 \quad , \qquad 0 < r_0 \le 1 \tag{25}$$

The same problem can be dealt with introduction of unreliability of the structure, defined as the probability of failure as follows

$$P_f = 1 - R \le p_0 \tag{26}$$

where p_0 is the level of unreliability which can be tolerated .When designing a structure the purpose is to keep the reliability as much as possible close at unity. If the random variable *E* is characterized through its probability density function, we can express some specific design parameter, in particular the length of the bar *L,* as depending on number of elements *N* and on the value of r_0.

Since in buckling circumstances we know the exact expression for the critical load it is possible to also evaluate the exact reliability.

We can, therefore, evaluate general expression for the “actual” reliability, according to parameters *N* and $r_{0,}$ that can be obtained substituting the parameter *L* deduced from approximate analysis in Eq.(24) into the expression of the exact value of critical load for a simply supported column.

Accuracy of FDM, in the stochastic setting, can be evaluate from the actual reliability values compared with the required r_0.

5. Example of the Exponentially Distributed Elasticity Modulus

To give a numerical example let us consider the case of a fixed distribution for the random modulus of elasticity, in particular an exponential distribution expressed by:

$$f_E(e) = \begin{cases} 0, & e < 0 \\ a\exp[-ae], & e \geq 0, a > 0 \end{cases}$$

Mathematical expectation and variance are respectively, $\mathrm{M}[E] = 1/a$ and $\mathrm{Var}[E]=1/a^2$. Keeping in mind Eq.(24) the approximate reliability takes the following form

$$R_{approx} = \exp\left[-\frac{1}{\mathrm{M}[E]}\frac{L^2 P_0}{\pi^2 I}\left(\frac{\pi/2N}{\sin \pi/2N}\right)^2\right] \qquad (27)$$

By demanding that R_{appox} equals its codified value r_0 , we obtain for the designed quantity, namely the length L of the column the following expression :

$$L_{approx} = L(N, r_0) = \pi\sqrt{\frac{I\mathrm{M}[E]}{P_0}\ln\frac{1}{r_0}}\left(\frac{\sin \pi/2N}{\pi/2N}\right) \qquad (28)$$

The exact expression for critical load of a simply supported end is given by $P_{cr} = \pi^2 EI/L^2$. We define the exact reliability as the following expression:

$$R_{exact} = Prob\left(\frac{\pi^2 EI}{L^2} \geq P_0\right) \qquad (29)$$

Substitution of approximate value for the length (Eq.27) in the expression of exact reliability allows to evaluate the actual reliability as follows:

$$R_{actual} = R_{actual}(N, r_0) = R_{exact}\big|_{L=L_{approx}} = 1 - \left[1 - \exp\left(-\frac{1}{\mathrm{M}[E]}\frac{P_0 L^2{}_{approx}}{\pi^2 I}\right)\right] \qquad (30)$$

or

$$R_{actual} = \exp\left[\left(\frac{\sin \pi/4N}{\pi/4N}\right)^2 \ln(r_0)\right] = r_0^{[(\sin \pi/4N)/(\pi/4N)]^2} \tag{31}$$

Evaluating R_{actual} for increasing number of N we obtain values that are always greater than r_0 or equal r_0.

In the Figures 1 the percentage errors between R_{actual} and r_0 for increasing value of N and for r_0 equal, respectively, to 0.90, 0.99, 0.999 and 0.9999 are depicted.

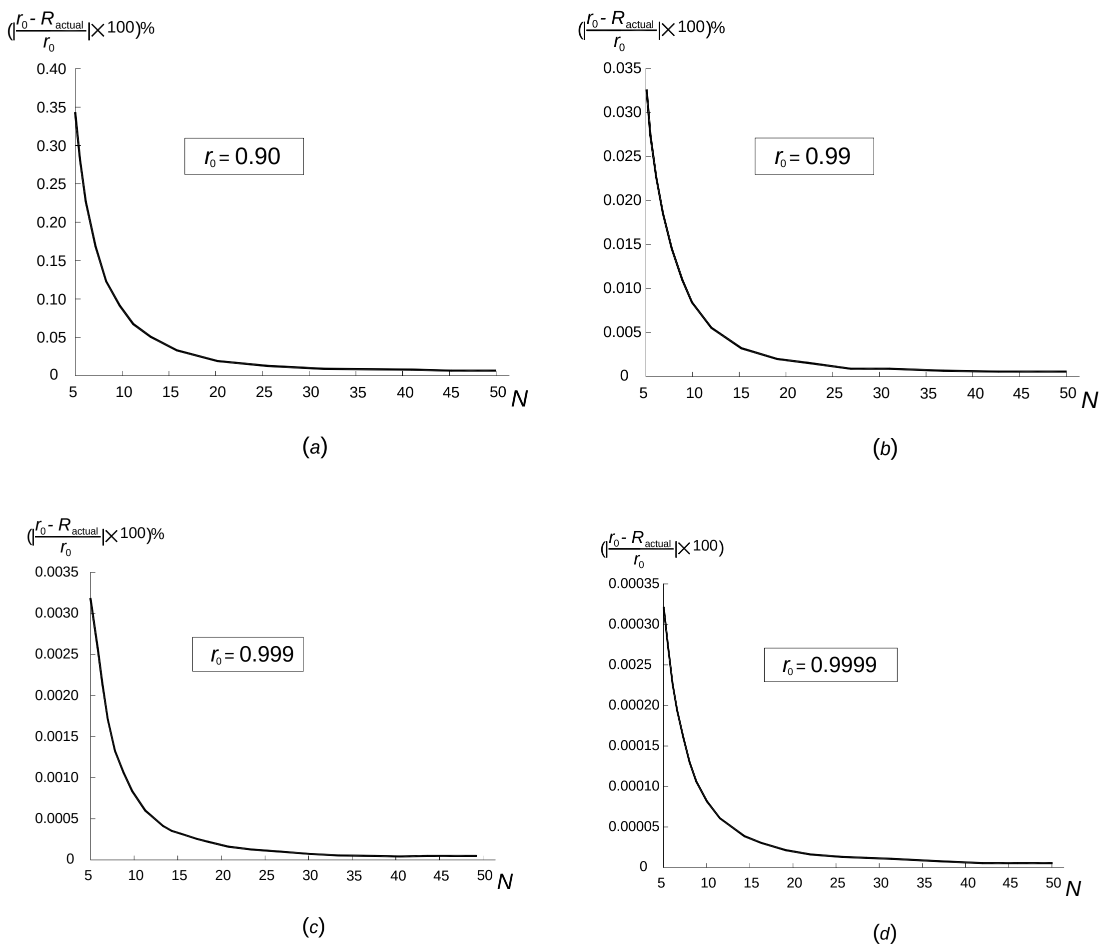

Fig. 1:Percentagewise difference between the codified and actual reliabilities

When the value of r_0 is fixed in 0.90 (Fig.1*a*) the percentage error goes from 0.343.% for N=5 (R_{actual}=0.903084) to 0.0864% for N=10 (R_{actual}=0.900778) to 0.0385% for N=15(R_{actual}=0.900346).

Keeping in mind the relation between reliability and probability of failure we can evaluate analogously the *actual* probability of failure for fixed values of the tolerated one.

Fixing p_0 equal respectively to 0.1, 0.01, 0.001 and 0.0001, the Fig.2 shows the percentage error between $P_{f,actual}$ and p_0 for increasing value of N.

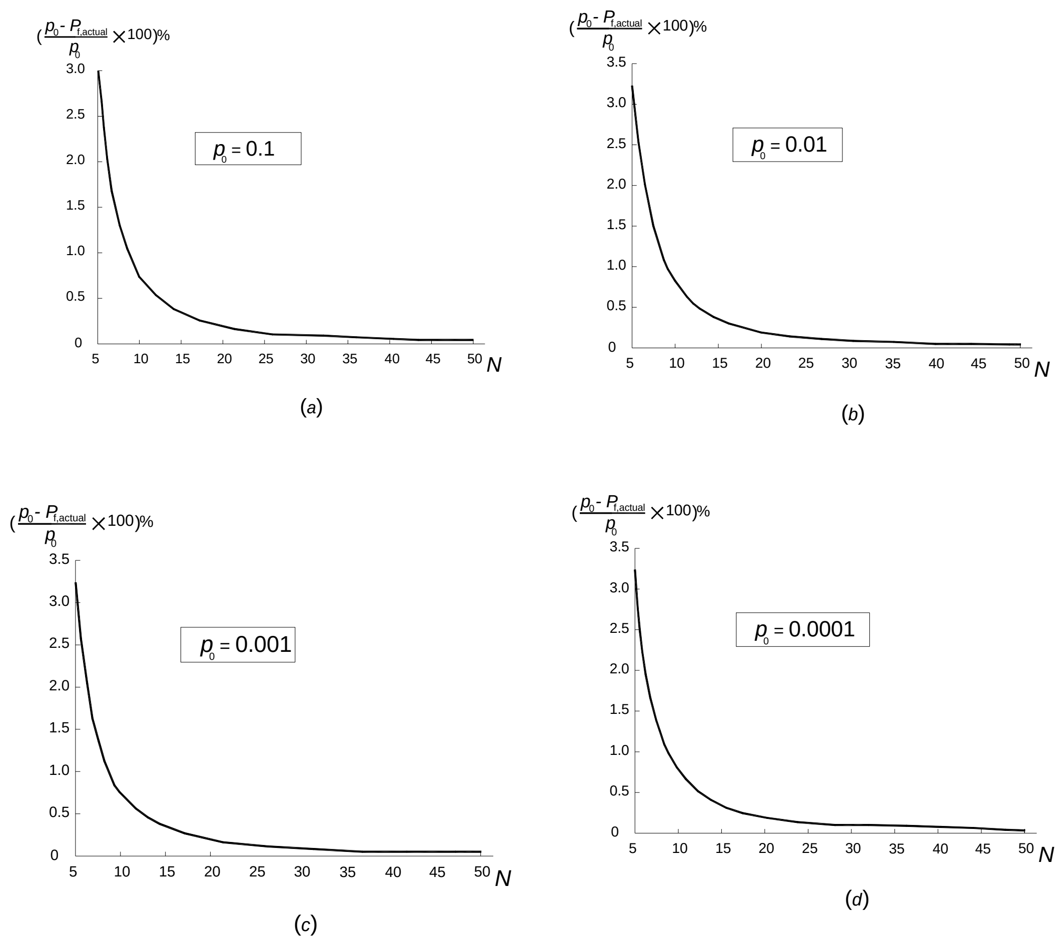

Fig.2: Error in probability of failure: Modulus of elasticity random variable with Exponential Distribution

Fixing the number of elements at N=5, we get $P_{f,actual}$=0.0969159 for p_0=0.1 (ε=3.08411%), $P_{f,actual}$=0.00967689 for p_0=0.01 (ε=3.23112%), $P_{f,actual}$=0.000967547 for p_0=0.001 (ε=3.245%) and $P_{f,actual}$=0.0000967533 for p_0=0.0001 (ε=3.24672%).

6. Conclusion

In this paper the reliability of the reliability calculations was studied in the buckling context. The question that was posed here is as follows: Is actual probability of failure greater than, equal to, or less than the tolerable probability of failure that is pertinent to the ideal, error-free situation?

In the example that is presented in this investigation the actual probability of failure turns out to be smaller than the tolerable level. This appears to be a good news for the reliability of the reliability calculation of the finite difference method. Whereas this conclusion cannot be extended to the other cases of the use of finite difference method, still it represents an interesting finding that was not anticipated *a priori* by the present investigators.

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