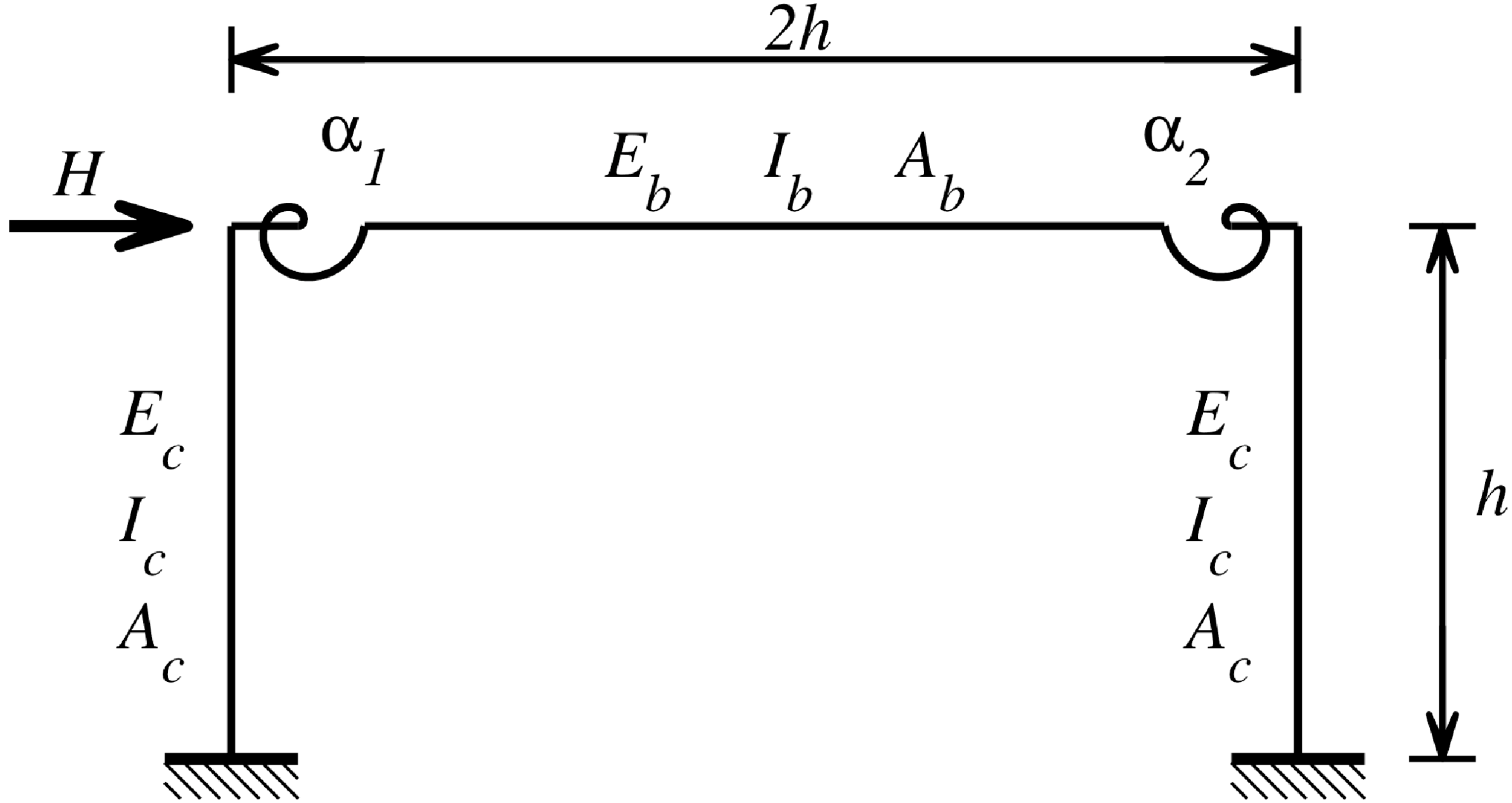
Formulation for Reliable Analysis of Structural Frames

George Corliss, Computer Engr., Marquette
Chris Foley, Civil Engr., Marquette
R. Baker Kearfott, Math, Louisiana

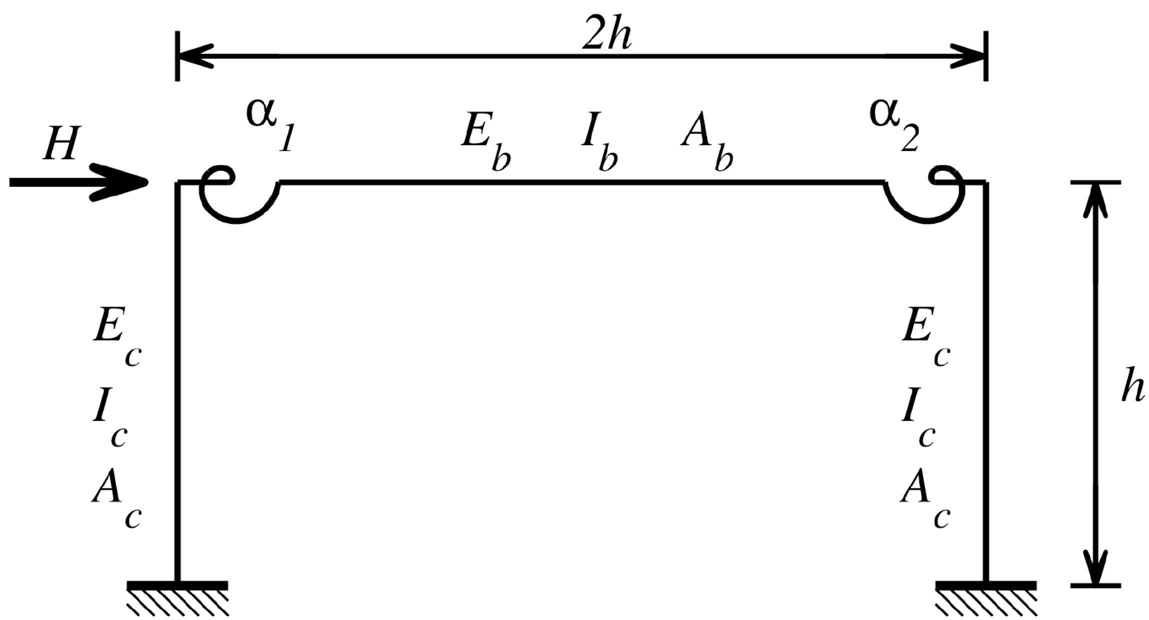
NSF Workshop on Reliable Engineering Computing
September 15-17, 2004, Savannah, Georgia



Abstract

Structural engineers use design codes formulated to consider uncertainty for both reinforced concrete and structural steel design. For a simple one-bay structural steel frame, we survey typical uncertainties and compute an interval solution for displacements and forces. The naive solutions have large over-estimations, so we explor the Mullen-Muhanna element-by-element strategy, scaling, and constraint propagation to achieve tight enclosures of the true ranges for displacements and forces in a fraction of the CPU time typically used for simulations. That we compute tight enclosures, even for larg parameter uncertainties used in practice, suggests the promise of interval methods for much larger structures.

Uncertain Parameters



Typical parameter values (note wide intervals):

$E_b = E_c =$ 29,000,000 $\pm$ 3,480,000 lbs/in^2 ($\pm$12%) (Young modulu

$I_b =$ 510 $\pm$ 51 in^4; $I_c =$ 272 $\pm$ 27.2 in^4 ($\pm$10%) (Second moment)

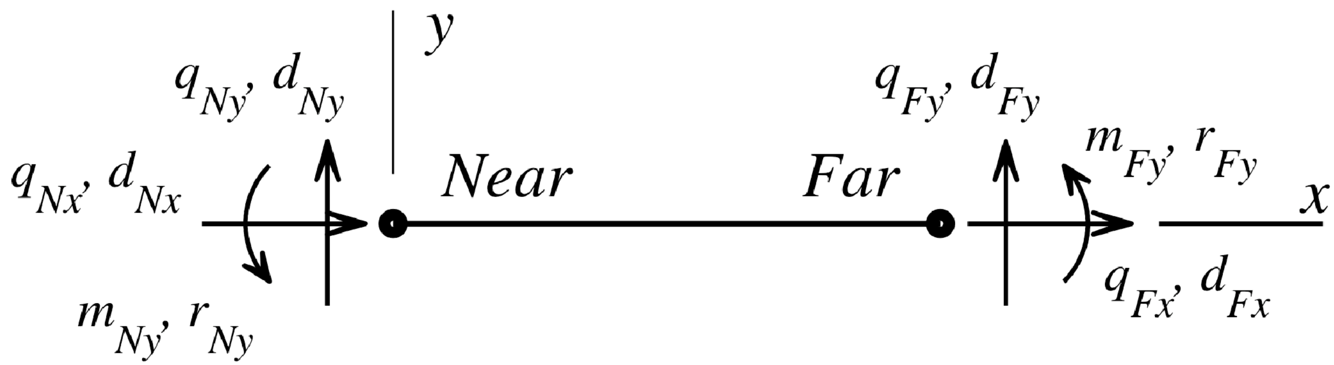
$A_b =$ 10.3 $\pm$ 10.3 in^2; $A_c =$ 14.4 $\pm$ 1.44 in^2 ($\pm$10%) (Area)

$H =$ 5,305.5 $\pm$ 2,203.5 lbs ($\pm$41.6%) (External force)

$\alpha =$ 277,461,000 $\pm$ 126,504,000 lb-in/rad ($\pm$45.6%) (Joint stiffne

$L_c =$ 144 in; $L_b = 2L_c$ (Length)

Component: Member

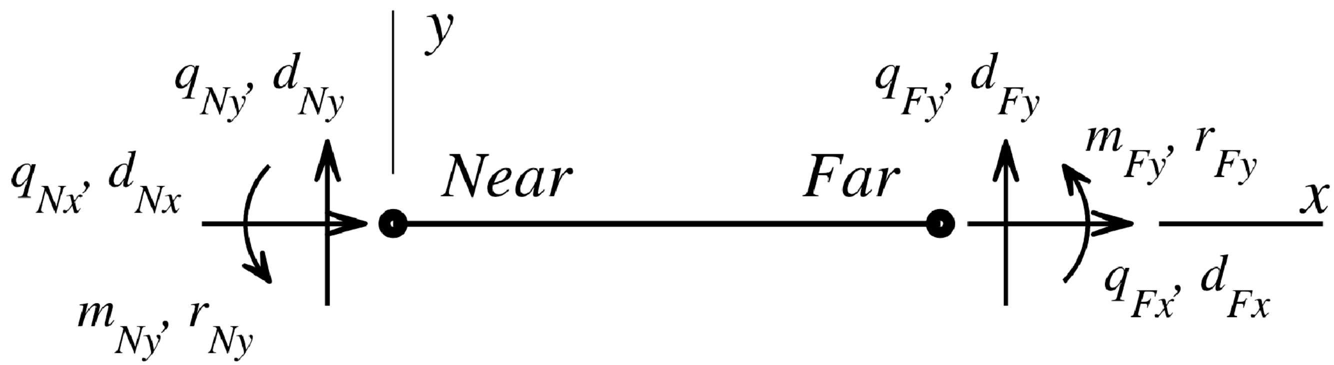


Attributes (in local $\widehat{(\cdot)}$ or global coordinates)*:

- Displacements: $d_{N\hat{x}}$, $d_{N\hat{y}}$, $d_{F\hat{x}}$, $d_{F\hat{y}}$ (local) or d_{Nx}, d_{Ny}, d_{Fx}, d_{Fy} (global)
- Rotations: $r_{N\hat{z}}$, $d_{F\hat{z}}$ (local) or r_{Nz}, d_{Fz} (global)
- Forces: $q_{N\hat{x}}$, $q_{N\hat{y}}$, $q_{F\hat{x}}$, $q_{F\hat{y}}$ (local) or q_{Nx}, q_{Ny}, q_{Fx}, q_{Fy} (global)
- Moments: $m_{N\hat{z}}$, $m_{F\hat{z}}$ (local) or m_{Nz}, m_{Fz} (global)

*Hibbeler, *Structural Analysis*, 2002

Component: Member



Properties: Frame-member stiffness equation:

$$\begin{bmatrix} \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ \frac{-AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} d_{N\hat{x}} \\ d_{N\hat{y}} \\ r_{N\hat{z}} \\ d_{F\hat{x}} \\ d_{F\hat{y}} \\ r_{F\hat{z}} \end{bmatrix} = \begin{bmatrix} q_{N\hat{x}} \\ q_{N\hat{y}} \\ m_{N\hat{z}} \\ q_{F\hat{x}} \\ q_{F\hat{y}} \\ m_{F\hat{z}} \end{bmatrix}$$

Local coordinates transformed to global coordinates

Component: End

"End" is an end of a Member or a Joint
Ends define the topology of the structure

Attributes (in global coordinates):

- Displacements: d_x, d_y; Rotations: r_z
- Forces: q_x, q_y, q_{Ex}, q_{Ey}; Moments: m_z, m_{Ez},

Properties:

- Can be incident with two or more Members and Joints
- Displacements d_x, d_y, and r_z are equal for all Members and Joints incident on an End
- Forces $q_{Ex} + q_x$, $q_{Ey} + q_y$, and moments $m_{Ez} + m_z$ each sum to zero for all Members and Joints incident on an End

Component: Joint

Attributes (in local or global coordinates):

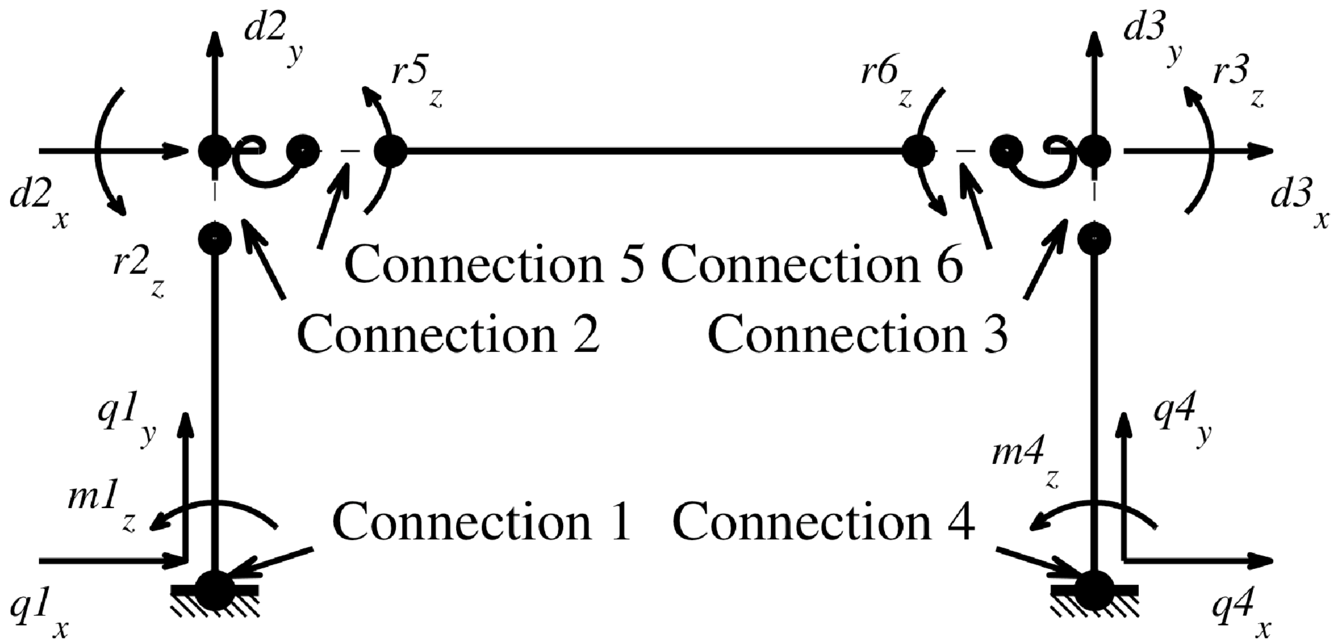
- Displacements: $d_{N\hat{x}}$, $d_{N\hat{y}}$, $d_{F\hat{x}}$, $d_{F\hat{y}}$; Rotations: $r_{N\hat{z}}$, $d_{F\hat{z}}$
- Forces: $q_{N\hat{x}}$, $q_{N\hat{y}}$, $q_{F\hat{x}}$, $q_{F\hat{y}}$; Moments: $m_{N\hat{z}}$, $m_{F\hat{z}}$

Properties:

- Length = 0
- Joins one Member to another
- Global displacements d_x are equal for incident End and Member
- Global displacements d_y are equal for incident End and Member
- Global forces q_x are equal for incident End and Member
- Global forces q_y are equal for incident End and Member
- Local rotations and moments satisfy

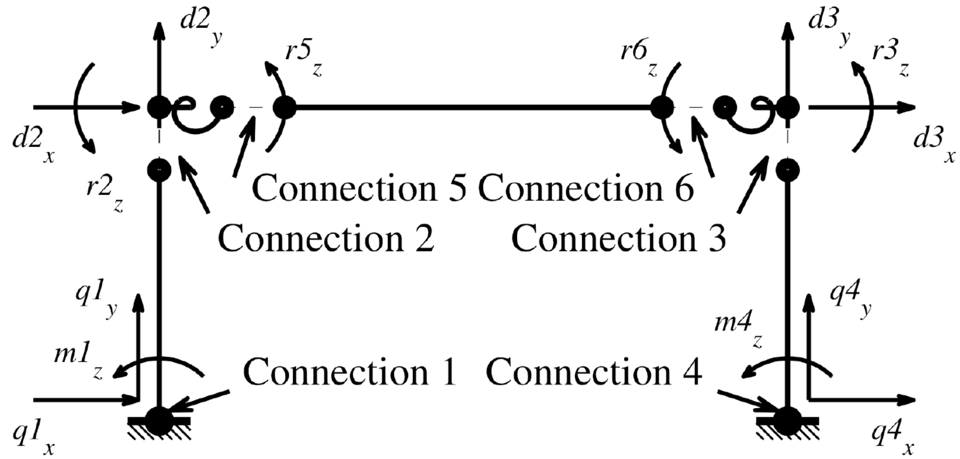
$$\begin{bmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{bmatrix} \begin{bmatrix} r_{N\hat{z}} \\ r_{F\hat{z}} \end{bmatrix} = \begin{bmatrix} m_{N\hat{z}} \\ m_{F\hat{z}} \end{bmatrix}$$

Assembly



$$
\begin{bmatrix}
\frac{12E_cI_c}{L_c^3}+\frac{A_bE_b}{L_b} & 0 & \frac{6E_cI_c}{L_c^2} & 0 & 0 & -\frac{A_bE_b}{L_b} & 0 & 0 \\
0 & \frac{A_cE_c}{L_c}+\frac{12E_bI_b}{L_b^3} & 0 & \frac{6E_bI_b}{L_b^2} & \frac{6E_bI_b}{L_b^2} & 0 & -\frac{12E_bI_b}{L_b^3} & 0 \\
\frac{6E_cI_c}{L_c^2} & 0 & \frac{4E_cI_c}{L_c}+\alpha & -\alpha & 0 & 0 & 0 & 0 \\
0 & \frac{6E_bI_b}{L_b^2} & -\alpha & \frac{4E_bI_b}{L_b}+\alpha & \frac{2E_bI_b}{L_b} & 0 & -\frac{6E_bI_b}{L_b^2} & 0 \\
0 & \frac{6E_bI_b}{L_b^2} & 0 & \frac{2E_bI_b}{L_b} & \frac{4E_cI_c}{L_c}+\alpha & 0 & -\frac{6E_bI_b}{L_b^2} & -\alpha \\
-\frac{A_bE_b}{L_b} & 0 & 0 & 0 & 0 & \frac{A_bE_b}{L_b}+\frac{12E_cI_c}{L_c^3} & 0 & \frac{6E_cI_c}{L_c^2} \\
0 & -\frac{12E_bI_b}{L_b^3} & 0 & -\frac{6E_bI_b}{L_b^2} & -\frac{6E_bI_b}{L_b^2} & 0 & \frac{12E_bI_b}{L_b^3}+\frac{A_cE_c}{L_c} & -\frac{6E_bI_b}{L_b^2} \\
0 & 0 & 0 & 0 & -\alpha & \frac{6E_cI_c}{L_c^2} & -\frac{6E_bI_b}{L_b^2} & \frac{4E_cI_c}{L_c}+\alpha
\end{bmatrix}
$$

Approximate Solution



Solve using mid-point values of parameters:

	Displacement d_x	Displacement d_y	Rotation r_z
Connection 2	0.15356843	0.00033236	-0.00096285
Connection 3	0.15102784	-0.00033236	-0.00094313
Connection 5			-0.00045995
Connection 6			-0.00044556
	Force q_x	**Force q_y**	**Moment m_z**
Connection 1	-2670.516	-963.856	245019.992
Connection 4	-2634.984	963.856	241381.602

8 $\times$ 8 system, condition number cond(K) = 4.7e+04

Naive Interval Solution

With uncertainties 1% of given:

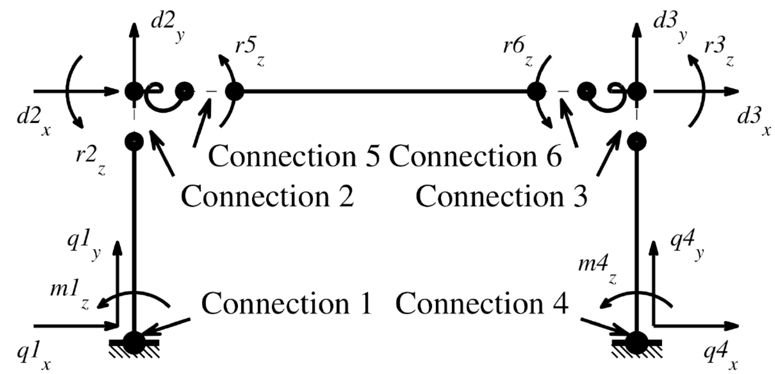
Disp.	Float	Interval True range	Midpoint ± Radius Rel. overest.
$d2_x$	0.153568	[0.09375783, 0.21337873]	0.1535683 ± 0.05981
	Tight:	[0.15237484, 0.15476814]	76.34%
$d2_y$e+3	0.332364	[0.19060424, 0.47412283]	0.3323635 ± 0.1418
	Tight:	[0.32940418, 0.33533906]	83.52%
$r2_z$e+3	-0.962852	[-1.3531968, -0.57250484]	-0.9628508 ± 0.3903
	Tight:	[-0.97085151, -0.95490139]	79.42%
$r5_z$e+3	-0.459955	[-0.6557609, -0.26414725]	-0.4599541 ± 0.1958
	Tight:	[-0.4638112, -0.45611532]	83.47%
$r6_z$e+3	-0.445563	[-0.64100045, -0.2501251]	-0.4455628 ± 0.1954
	Tight:	[-0.44930811, -0.4418354]	86.05%
$d3_x$	0.151028	[0.091230936, 0.21082444]	0.1510277 ± 0.0598
	Tight:	[0.14985048, 0.15221127]	77.62%
$d3_y$e+3	-0.332364	[-0.47412283, -0.19060424]	-0.3323635 ± 0.1418
	Tight:	[-0.33533906, -0.32940418]	83.52%
$r3_z$e+3	-0.943133	[-1.3330326, -0.55323186]	-0.9431322 ± 0.3899
	Tight:	[-0.95100335, -0.93531196]	81.02%

Interval solutions are hopelessly pessimistic

At 4% of given uncertainties, stiffness matrix includes singular matrix

Element-By-Element

Mullen & Muhanna (1999):
Introduce extra variables and add extra equations to the system to reduce the interval dependencies.
Each Member & Joint is separate



Member M_1 global stiffness:

$$\begin{bmatrix} \frac{12E_cI_c}{L_c^3} & 0 & -\frac{6E_cI_c}{L_c^2} & -\frac{12E_cI_c}{L_c^3} & 0 & -\frac{6E_cI_c}{L_c^2} \\ 0 & \frac{A_cE_c}{L_c} & 0 & 0 & -\frac{A_cE_c}{L_c} & 0 \\ -\frac{6E_cI_c}{L_c^2} & 0 & \frac{4E_cI_c}{L_c} & \frac{6E_cI_c}{L_c^2} & 0 & \frac{2E_cI_c}{L_c} \\ -\frac{12E_cI_c}{L_c^3} & 0 & \frac{6E_cI_c}{L_c^2} & \frac{12E_cI_c}{L_c^3} & 0 & \frac{6E_cI_c}{L_c^2} \\ 0 & -\frac{A_cE_c}{L_c} & 0 & 0 & \frac{A_cE_c}{L_c} & 0 \\ -\frac{6E_cI_c}{L_c^2} & 0 & \frac{2E_cI_c}{L_c} & \frac{6E_cI_c}{L_c^2} & 0 & \frac{4E_cI_c}{L_c} \end{bmatrix} \begin{bmatrix} d1_x \\ d1_y \\ r1_z \\ d2_x \\ d2_y \\ r2_z \end{bmatrix} - \begin{bmatrix} q1_x \\ q1_y \\ m1_z \\ q2_x \\ q2_y \\ m2_z \end{bmatrix} = 0$$

60×60 system with $\text{Cond}(K) = 1.2\text{e}{+}17$
Intervals essentially the same as naive interval solution

Subdistributivity

In interval arithmetic: $a(b + c) \subseteq ab + ac$ (subdistributivity). E.g.,

$$[-1, 2] * ([4, 5] + [-3, -2]) = [-3, 6] \subseteq [-1, 2] * [4, 5] + [-1, 2] * [-3, -2$$

To get tighter enclosures, factor (as Mullen & Muhanna)

Member M_1 global stiffness matrix:

$$\begin{bmatrix} \frac{12E_cI_c}{L_c^3} & 0 & -\frac{6E_cI_c}{L_c^2} & -\frac{12E_cI_c}{L_c^3} & 0 & -\frac{6E_cI_c}{L_c^2} \\ 0 & \frac{A_cE_c}{L_c} & 0 & 0 & -\frac{A_cE_c}{L_c} & 0 \\ -\frac{6E_cI_c}{L_c^2} & 0 & \frac{4E_cI_c}{L_c} & \frac{6E_cI_c}{L_c^2} & 0 & \frac{2E_cI_c}{L_c} \\ -\frac{12E_cI_c}{L_c^3} & 0 & \frac{6E_cI_c}{L_c^2} & \frac{12E_cI_c}{L_c^3} & 0 & \frac{6E_cI_c}{L_c^2} \\ 0 & -\frac{A_cE_c}{L_c} & 0 & 0 & \frac{A_cE_c}{L_c} & 0 \\ -\frac{6E_cI_c}{L_c^2} & 0 & \frac{2E_cI_c}{L_c} & \frac{6E_cI_c}{L_c^2} & 0 & \frac{4E_cI_c}{L_c} \end{bmatrix} \begin{bmatrix} d1_x \\ d1_y \\ r1_z \\ d2_x \\ d2_y \\ r2_z \end{bmatrix} - \begin{bmatrix} q1_x \\ q1_y \\ m1_z \\ q2_x \\ q2_y \\ m2_z \end{bmatrix} =$$

Subdistributivity

Let

$$d_{63} := \frac{A_c E_c}{L_c}(d1_y - d2_y) \qquad d_{65} := \frac{2E_c I_c}{L_c}(r1_z + r2_z)$$
$$d_{64} := \frac{6E_c I_c}{L_c^2}(d1_x - d2_x) \qquad d_{66} := \frac{2E_c I_c}{L_c} r1_z$$

Leads to a considerably simpler system:

$$\frac{2}{L_c}d_{64} - \frac{3}{L_c}d_{65} - q1_x = 0; \quad d_{63} - q1_y = 0$$
$$-d_{64} + d_{65} + d_{66} - m1_z = 0$$
$$q1_x + q2_x = 0; \quad q1_y + q2_y = 0$$
$$-d_{64} + 2d_{65} - d_{66} - m2_z = 0$$
$$d1_y - d2_y - \frac{L_c}{A_c E_c}d_{63} = 0; \quad d1_x - d2_x - \frac{L_c^2}{6E_c I_c}d_{64} = 0$$
$$r1_z + r2_z - \frac{L_c}{2E_c I_c}d_{65} = 0; \quad r1_z - \frac{L_c}{2E_c I_c}d_{66} = 0$$

Element-by-Element Interval Solution

Disp.	Float	Interval True range	Midpoint ± Radius Rel. overest.
$d2_x$	0.153568	[0.15206288, 0.15507492]	0.1535689 ± 0.001506
	Tight:	[0.15237484, 0.15476814]	0.40%
$d2_y$e+3	0.332364	[0.32918317, 0.33554758]	0.3323654 ± 0.003182
	Tight:	[0.32940418, 0.33533906]	0.13%
$r2_z$e+3	-0.962852	[-0.97485786, -0.95084958]	-0.9628537 ± 0.012
	Tight:	[-0.97085151, -0.95490139]	0.84%
$r5_z$e+3	-0.459955	[-0.46757208, -0.45234116]	-0.4599566 ± 0.007615
	Tight:	[-0.4638112, -0.45611532]	1.63%

74 × 74 system, Cond(K) = 1.2E+17

Intervals are MUCH tighter

Scaling

Reduce Cond(K) = 1.2e+17 by scaling?

Replace force H by its midpoint $\tilde{H}$; multiply solution by $[H]/\tilde{H}$
Replace each variable force ($q1_x$, ...) by force/$\tilde{H}$
Eliminate known "variables"

Interval solution using scaled element-by-element approach with 1%
given uncertainties

Disp.	Float	Interval True range	Midpoint ± Radius Rel. overest.
$d2_x$	0.153568	[0.15294597, 0.15419182]	0.1535689 ± 0.0006229
	Tight:	[0.1531698, 0.15396904]	0.29%
$d2_y$e+3	0.332364	[0.33111682, 0.33361393]	0.3323654 ± 0.001249
	Tight:	[0.3311227, 0.33360764]	0.004%
$r2_z$e+3	-0.962852	[-0.96945816, -0.95624927]	-0.9628537 ± 0.006604
	Tight:	[-0.96583881, -0.95988319]	0.75%
$r5_z$e+3	-0.459955	[-0.46515166, -0.45476159]	-0.4599566 ± 0.005195
	Tight:	[-0.46141645, -0.45849491]	1.62%

Scaling

Interval solution using scaled element-by-element approach with GIVEN uncertainties

Disp.	Float	Interval True range	Midpoint ± Radius Rel. overest.
$d2_x$	0.153568	[0.022924888, 0.29366922]	0.1582971 ± 0.1354
	Tight:	[0.12130751, 0.20804041]	119.8%
$d2_y$e+3	0.332364	[0.11407836, 0.57891094]	0.3464947 ± 0.2324
	Tight:	[0.21526742, 0.47234932]	62.51%
$r2_z$e+3	-0.962852	[-2.5286276, 0.55857565]	-0.985026 ± 1.544
	Tight:	[-1.4124783, -0.73502904]	250.3%
$r5_z$e+3	-0.459955	[-1.7359689, 0.77691797]	-0.4795255 ± 1.256
	Tight:	[-0.6216869, -0.3157181]	479.8%

58 × 58 system, Cond(K) = 2.8e+08

Successful solution with 1.7 times given uncertainties

+ Interval arithmetic can handle realistic uncertainties

– Enormous overestimations

Constraint Propagation

Find roots in $[-4, 4]$ of $f(x) = x^2 + x - 5 = 0$

Solve for $x = 5 - x^2$
Substitute $x = [-4, 4]$: $x = 5 - [-4, 4]^2 = [-11, 5]$
Not especially helpful

Solve for $x = \pm\sqrt{5 - x}$
Substitute $x = [-4, 4]$: $x = \pm\sqrt{5 - [-4, 4]} = [-3, -1] \cup [1, 3]$
Iterate $x = \sqrt{5 - x}$ from $x = [1, 3]$:

$$
\begin{aligned}
x &= [1.41421356237309, 2.00000000000000] \\
& [1.73205080756887, 1.89361728911280] \\
& [1.76249332222485, 1.80774699347866] \\
& [1.78668771936265, 1.79930727719730]
\end{aligned}
$$

converging to the root $x^* = 1.79128784747792$.

Constraint Propagation

From logic programming, see Van Hentenryck et al., *Numerica: A Modeling Language for Global Optimization*, MIT, 1997

Constraint propagation:

Discard candidate solutions infeasible with respect to already known information

Gauss-Seidel, except solve each equation for each variable

Especially attractive for sparse systems, such as ours

Constraint Propagation

Start with $[0.6, 1.4]$ times the approximate solution using midpoints
Use **no** interval system solver
Five iterations of constraint propagation

Disp.	Float	Interval True range	Midpoint ± Radius Rel. overest.
$d2_x$	0.153568	[0.076784216, 0.23035265]	0.1535684 ± 0.07678
	Tight:	[0.12130751, 0.20804041]	43.52%
$d2_y$e+3	0.332364	[0.166182, 0.49854599]	0.332364 ± 0.1662
	Tight:	[0.21526742, 0.47234932]	22.65%
$r2_z$e+3	-0.962852	[-1.4442773, -0.48142577]	-0.9628515 ± 0.4814
	Tight:	[-1.4124783, -0.73502904]	29.64%
$r5_z$e+3	-0.459955	[-0.68993206, -0.22997735]	-0.4599547 ± 0.23
	Tight:	[-0.6216869, -0.3157181]	33.48%

We achieve relative over-estimations that are comparable to the relative uncertainties in the parameters, in spite of a condition numb of 2.8e+08

Conclusions: To Structural Engineers

- Reliable interval techniques are feasible
- One set of interval computations can guarantee to enclose the displacements, rotations, forces, and moments that could be observed from any combination of values of cross-sectional properties, loading, material properties, and connections, even for quite large uncertainties.
- Alternative to Monte Carlo simulations

Next: Extend to non-linear behaviors and to larger structures

Conclusions: To Interval Researchers

- Variety of techniques to achieve relatively tight enclosures of the solution to a realistic (although small) problem, even in the face parameter uncertainties over 40%
- Element-by-element adds equations specifying that two variables are the same, rather than simplifying by identifying them with the same variable.
- Used symbolic rearrangement, scaling of the equations, and constraint propagation

Next: larger systems, sparsity-preserving preconditioning, more effective and efficient constraint propagation, and branch-and-bound-like strategies for subdividing the ranges of wide interval-valued parameters