# Improved Methods for Solutions of Complentarity Problems

by

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# Contents

Find $z \in \mathbb{R}^n$ such that for $l : \mathbb{R}^n \to \mathbb{R}^n$

$$\left.\begin{aligned} z &\geq 0 \\ l(z) &\geq 0 \\ z^T l(z) &= 0 \end{aligned}\right\} \begin{array}{l} \text{Nonlinear} \\ \text{Complementarity} \\ \text{Problem (NCP)} \end{array}$$

(or show that no such $z$ exist)

## Equivalent formulations:

Find $w, z \in \mathbb{R}^n$ such that

$$\left.\begin{array}{r} w \geq 0,\ z \geq 0 \\ w = l(z) \\ z^T w = 0 \end{array}\right\} \text{(NCP)}$$

$$\left.\begin{array}{l} \text{Find } z \geq 0 \text{ such that} \\ g(z) = min(z, l(z)) = 0 \end{array}\right\} \text{(NCP)}$$

If $l(z) = Mz + q$, $M \in \mathbb{R}^{n\times n}$, $q \in \mathbb{R}^n$:
"Linear Complementarity Problem" (LCP)

# 2. Examples

**Example:** Quadratic Programming (QP)

minimize

$$f(x) = c^T x + \frac{1}{2} x^T Q x$$

subject to

$$\begin{aligned} Ax &\geq b \\ x &\geq 0. \end{aligned}$$

$Q \in \mathbb{R}^{n \times n}$ symmetric, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

$Q = 0 \Rightarrow (LP)$

$x$ locally optimal solution $\Rightarrow \exists y \in \mathbb{R}^n$ s. t. $(x, y)^T$ satisfies the Karush-Kuhn-Tucker conditions

$$\left.\begin{array}{ll} u = c + Qx - A^T y \geq 0 & , x \geq 0,\ x^T u = 0 \\ v = -b + Ax \geq 0 & , y \geq 0,\ y^T v = 0 \end{array}\right\} (*)$$

$$M = \begin{pmatrix} Q & -A^T \\ A & 0 \end{pmatrix} \quad q = \begin{pmatrix} c \\ -b \end{pmatrix} \quad z = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{cases} Mz + q \geq 0 \\ z^T(q + Mz) = 0 \quad \text{(LCP)} \\ z \geq 0 \end{cases}$$

**Remark:** If $Q$ is p. d. (i. e. $f(x)$ is convex) than $(*)$ is sufficient for $x$ to be a globally optimal solution.

Further problems which lead to Complementarity problems:

Contact problem
Porous flow problem
Obstacle problem
Journal bearing problem
Elastic plastic torsion problem
⋮

R. W. Cottle et al. , The linear complementarity problem. Academic Press 1991

**Example**

$$l(z) = Mz + q,\ M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix},\ q = \begin{pmatrix} 2 \\ 1 \\ -10^{-6} \end{pmatrix}$$

Find $w, z \geq 0$ s. t.

$$\begin{aligned} w &= l(z) \\ z^T w &= 0\,. \end{aligned}$$

For the "approximate solution"

$$z = \begin{pmatrix} 10^{-6} \\ 10^{-6} \\ 1 \end{pmatrix},\ w = \begin{pmatrix} 3 \\ 2 \\ 10^{-6} \end{pmatrix}$$

one obtains

$$\begin{aligned} &\|l(z) - w\|_\infty = \|Mz + q - w\|_\infty = 4 \cdot 10^{-6} \\ &z^T w = 6 \cdot 10^{-6} \end{aligned}$$

Stopping criteria (e. g. for interior point method))

$$\max \{z^T w, \|Mz + q - w\|_\infty\} \le \varepsilon$$

"$\varepsilon$-approximate solution"
$(z, w)$ is a $6 \cdot 10^{-6}$ - approximate solution

**However:**

$[x] = z + r[-e, e]\,,\, r = 0.25, \Rightarrow$
$[x] \cap L(z, A, [x]) = \emptyset$

$\Rightarrow$ There is no exact solution within an $l_\infty$ distance of 0.25 from an $\varepsilon$-approximate solution with $\varepsilon = 6 \cdot 10^{-6}$.

# 3. Verification of Solutions

$$[x] \subseteq D \subset \mathbb{R}^n$$

$H : D \subset \mathbb{R}^n \to \mathbb{R}^n$, continuous;

$$H(x) - H(y) = \underbrace{\delta H(x,y)}_{\text{"slope"}}(x-y)\,,\ x, y \in [x]$$

Assume

$\delta H(x,y) \in \delta H(x,[x])$ for all $y \in [x]$
and some fixed $x \in [x]$

Define for $a$ nonsingular $A$

$$L(x, A, [x]) := x - A^{-1}H(x) + (I - A^{-1} \cdot \delta H(x,[x]))([x] - x)\,.$$

Then:

a) $L(x, A, [x]) \subseteq [x] \Rightarrow \exists \overset{*}{x} \in [x] : H(\overset{*}{x}) = 0$

b) $L(x, A, [x]) \cap [x] = \emptyset \Rightarrow H(x) \neq 0,\ x \in [x]$

(Brouwer fixed point theorem)

$$\left.\begin{array}{r}\text{Find } z \geq 0 \text{ such that} \\ g(z) = min(z, l(z)) = 0\end{array}\right\} \text{(NCP)}$$

1. $g(x) - g(y) = \delta g(x,y)(x-y)$

2. $\delta g(x,y) \in \delta g(x,[x])$ , $x \in [x]$ fixed for all $y \in [x]$

**Ad 1.**

$$S_i^+ = \{x \in [x] \mid l_i(x) > x_i\}$$

$$S_i^- = \{x \in [x] \mid l_i(x) < x_i\}$$

$$S_i^0 = \{x \in [x] \mid l_i(x) = x_i\}$$

$$g_i(x) = \begin{cases} x_i & , \quad x \in S_i^+ \\ l_i(x) & , \quad x \in S_i^- \\ l_i(x) = x_i & , \quad x \in S_i^0 \end{cases}$$

| $x \backslash y$ | $S_i^+$ | $S_i^-$ | $S_i^0$ |
|---|---|---|---|
| $S_i^+$ | $e_i^T$ | $\alpha_i(\delta l_i(x,y) - e_i^T) + e_i^T$ | $e_i^T$ |
| $S_i^-$ | $\beta_i(\delta l_i(x,y) - e_i^T) + e_i^T$ | $\delta l_i(x,y)$ | $\delta l_i(x,y)$ |
| $S_i^0$ | $e_i^T$ | $\delta l_i(x,y)$ | $e_i^T$ |

where

$$l_i(x) - l_i(y) = \delta l_i(x,y)(x-y)$$

$$\alpha_i = \frac{y_i - l_i(y)}{(\delta l_i(x,y) - e_i^T)(x-y)}$$

$$\beta_i = \frac{l_i(x) - x_i}{(\delta l_i(x,y) - e_i^T)(x-y)}$$

$$\Rightarrow g_i(x) - g_i(y) = \delta g_i(x,y)(x-y)$$

$$\Rightarrow \delta g(x,y)$$

**Proof.**

Let $x \in S_i^+$, $y \in S_i^-$. Then

$$
\begin{aligned}
g_i(x) - g_i(y) &= x_i - l_i(y) = y_i - l_i(y) + e_i^T(x-y) = \\
&= \frac{(y_i - l_i(y))(\delta l_i(x,y) - e_i^T)(x-y)}{(\delta l_i(x,y) - e_i^T)(x-y)} + e_i^T(x-y) \\
&= \underbrace{(\alpha_i(\delta l_i(x,y) - e_i^T) + e_i^T)}_{\delta g_i(x,y)}(x-y)
\end{aligned}
$$

Similar for the other cases.

**Ad 2.**

Let $x \in [x]$ fixed and consider the nonlinear programming problems

$$\begin{array}{ll} \text{min} & y_i - l_i(y) \\ \text{s. t.} & y \in [x] \end{array}$$

and

$$\begin{array}{ll} \text{max} & y_i - l_i(y) \\ \text{s. t.} & y \in [x] \end{array}$$

Let $y^{i,1}$ and $y^{i,2}$ be solutions of these problems.

Let

$$\alpha_i = \frac{y^{i,2} - l_i(y^{i,2})}{(\delta l_i(x, y^{i,2}) - e_i^T)(x - y^{i,2})}$$

$$\beta_i = \frac{l_i(x) - x_i}{(\delta l_i(x, y^{i,1}) - e_i^T)(x - y^{i,1})}$$

Then

$$g(x, y) \in g(x, [x])$$

if

$$
\delta g_i(x,[x]) = \begin{cases} e_i^T\,, & y^{i,2} \in S_i^+ \cup S_i^0 \\[2ex] \delta l_i(x,[x])\,, & y^{i,1} \in S_i^- \cup S_i^0 \\[2ex] [0,\alpha_i](\delta l_i(x,[x]) - e_i^T) + e_i^T\,, & \begin{array}{r} x \in S_i^+ \cup S_i^0, \\ y^{i,2} \in S_i^- \end{array} \\[4ex] [\beta_i,1](\delta l_i(x,[x]) - e_i^T) + e_i^T\,, & \begin{array}{r} x \in S_i^-, \\ y^{i,1} \in S_i^+ \end{array} \end{cases}
$$

**Proof.** ...

**Algorithm:**

Let $r > 0$ be a given tolerance and let $x$ be an approximate solution of

$$g(z) = min(z, l(z)) = 0\,.$$

Define

$$[x] = (x + r[-e, e]) \cap \mathbb{R}_+$$

where $e = (1, 1, \ldots, 1)^T$.

Choose a nonsingular $A$ and compute $L(x, A, [x])$.

If $L(x, A, [x]) \subseteq [x]$ then $g(\overset{*}{z}) = 0,\ \overset{*}{z} \in [x]$.

If $L(x, A, [x]) \cap [x] = \emptyset$ then $g(z) \neq 0,\ z \in [x]$.

Alefeld, G., Chen, X. and Potra, F.: Validation of Solutions to Complementarity Problems, Numerische Mathematik 83 (1999), pp. 1-23.

Alefeld, G., Chen, X. and Potra, F.: Numerical Validation of Solutions of Complementarity Problems: The Nonlinear Case, Numerische Mathematik 92 (2002), pp. 1-16.

# 4. Special Problems

(Nonlinear) Complementarity Problems

$$(1) \quad \begin{cases} z & \geq & 0 \\ l(z) & \geq & 0 \\ z^T l(z) & = & 0 \end{cases} \qquad\qquad l : \mathbb{R}^n \to \mathbb{R}^n$$

Find $z^* \in \mathbb{R}^n$ such that (1) holds (or prove that no such $z^*$ exists!).

Let $D = \text{diag } (d_i), \qquad d_i > 0, i = 1, 2, \ldots n.$

$z^*$ solves (1) iff $z^*$ is a fixed point of

$$p(x) = \max\{0, x - Dl(x)\}.$$

<u>Theorem 1</u> Let $[x] = ([x_i])$ be an interval vector. Let $l'([x])$ denote the interval arithmetic evaluation of $l'$ over $[x]$.

If

$$(2)\ \Gamma(x, [x], D) := \max\{0, x - D \cdot l(x) + (I - Dl'([x]))([x] - x)\} \subseteq [x]$$

where $x \in [x]$ is fixed, then there is a solution $z^*$ of (1) in $\Gamma(x, [x], D)$ (and therefore in $[x]$). All solutions of (1) are containd in $\Gamma(x, [x], D)$.

<u>Remark:</u> $\max\ \{0, [x]\} = [\max\{0, \underline{x}\}, \max\{0, \overline{x}\}]$

From the last statement in Theorem 1 it follows:

Corollory:

If $\Gamma(x, [x], D) \cap [x] = \emptyset$ then there is no solution of (1) in $[x]$.

The linear case for an H-matrix $M$:

$$l(z) = Mz + q$$

$M$ is an H-matrix iff there exists a $d = (d_i) \in \mathbb{R}^n, d_i > 0$ ( a "positive vector") such that

$$\sum_{j \neq i} |m_{ij}| d_j < |m_{ii}| d_i \ , \quad i = 1, 2, \ldots, n$$

Equivalently: Let $\tilde{M} = (\tilde{m}_{ij})$ ("comparison matrix")

with $\quad \tilde{m}_{ij} = \begin{cases} |m_{ii}| & if \quad i = j \\ -|m_{ij}| & if \quad i \neq j \end{cases}$

$M$ is called an H-matrix iff there exists a positive vector $d$ such that $\tilde{M} d > 0$.

Remarks

a) $A$ (real) H-matrix has positive or negative diagonal elements.

b) If the diagonal dements of a (real) H-matrix are all positive, then this matrix belongs to the set of P-matrices.

c) If $M$ is a P-matrix then the LCP-problem (1) has a unique solution for each $q$.

Let $M$ be an H-matrix with positive Diagonalelements $m_{ii}, i = 1, 2, \dots n$.
Set $\tilde{D}^{-1} = diag(m_{ii})$.

Then the test (2) reads:

(2*) $\Gamma(x, [x], D) = max\{0, x - D(Mx + q) + (I - DM)([x] - x)\} \subseteq [x]$.

Assume that (2*) is true for some given $[x]$.
Define $[x^0] := [x]$ and

(3) $[x]^{k+1} = \Gamma(x^k, [x^k], D) \cap [x^k]$
where $x^k = m([x^k])$ ("the center" of $[x^k]$).

Theorem 3 Let $M$ be an H-Matrix with positive diagonal elements. Set $D^{-1} = diag(m_{ii})$.

a) If the (unique) solution $z^*$ of (1) is contained in $[x^0]$ then the iteration method (3) is well-defined and $\lim_{k\to\infty} [x^k] = z^*$.

b) If $z^* \notin [x^0]$ then the intersection becomes empty after a finite number of steps.

Computing an $[x^0]$ such that $z^* \in [x^0]$:

$$D^{-1} = diag(m_{ii})$$

Define $u = (u_i)$ with

$$u_i = \begin{cases} 0 & if \quad q_i \geq 0, \\ -q_i & if \quad q_i < 0. \end{cases}$$

Solve $\tilde{M}d = u$

Then

$$z^* \in [x^0] := max\{0, -Dq + [-d + Du, d - Du]$$

## Example (Random test problems)

$M = (m_1, m_2, m_3, m_4)$, where

$$m_1 = \begin{pmatrix} 1.388713122168711 \\ -4.699766249426920e-1 \\ 7.370559770214220e-2 \\ -4.110090461033111e-1 \end{pmatrix},$$

$$m_2 = \begin{pmatrix} -4.699766249426920e-1 \\ 1.453401598450949 \\ 3.334909523505895e-2 \\ -5.175564143615730e-1 \end{pmatrix},$$

$$m_3 = \begin{pmatrix} 7.370559770214220e-2 \\ 3.334909523505895e-2 \\ 6.604515405730874e-1 \\ -1.651162344083680e-1 \end{pmatrix},$$

$$m_4 = \begin{pmatrix} -4.110090461033111e-1 \\ -5.175564143615730e-1 \\ -1.651162344083680e-1 \\ 1.47737356490058 \end{pmatrix}.$$

The matrix is diagonally dominant, and so it is an H-matrix. Furthermore, the diagonal elements are positive.

We choose

$$q = \begin{pmatrix} 9.252128641303051 \\ 2.789538442487311 \\ 9.950524251712144 \\ -3.325681126317601 \end{pmatrix},$$

and

$$q = \begin{pmatrix} 8.679035675427925e-1 \\ 2.692546385763099 \\ -1.549159013124430 \\ -2.845459307376360 \end{pmatrix},$$

respectively.

Table 1

| $i$ | $[enclosure]$ |
|---|---|
| 1 | [0.0000000000000, 0.0000000000000] |
| 2 | [0.0000000000000, 0.0000000000000] |
| 3 | [0.0000000000000, 0.0000000000000] |
| 4 | [1.0047976566298, 3.4973556312525] |

Table 2

| $i$ | $[enclosure]$ |
|---|---|
| 1 | [0.0000000000000, 1.2683905383166] |
| 2 | [0.0000000000000, 0.0984999233387] |
| 3 | [1.1528398968364, 3.5383718513568] |
| 4 | [0.3203206580309, 3.5317305428024] |

Table 3

| $Example$ | NUM1 | RAD1 | FUN1 |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | 21 | 4.4409e-16 | 0 |

Enclosing Solutions of Linear Complementarity Problems for H-matrices (with Z. Wang and Z. Shen). To appear in Reliable Computing.

Schäfer, U.: An Enclosure Method for Free Boundary Problems based on a Linear Complementarity Problem with Interval Data. Numer. Funct. Anal. Optim. 22(201), pp.991-1011

# 5. Interval Data

(LCP):

Given: $M \in \mathbb{R}^{n \times n}, \ q \in \mathbb{R}^n$

Find: $z \in \mathbb{R}^n, \ z \geq 0, \ q + Mz \geq 0, \ z^T(q + Mz) = 0$

Equivalent: Solve $l(z) = min \ \{z, \ q + Mz\} = 0$

Generalization:

Given: $[q] \in \mathbb{IR}^n, [M] \in \mathbb{IR}^{n\times n}$

$$\sum := \{z \in \mathbb{R}^n | z \geq 0,\ q + Mz \geq 0,\ z^T(q + Mz) = 0,\ M \in [M],\ q \in [q]\}$$

Find: $[x] \in \mathbb{IR}^n$ such that $\sum \subseteq [x]$.

## Theorem 1:

Let $[M] = ([m_{ij}])$ be an H-matrix with $\underline{m}_{ii} > 0$, $i = 1, 2, \ldots, n$.

Let $[M] = [D] - [R]$ where

$$[D] = \text{diag } ([m_{ij}]),\ \ [R] = ([r_{ij}]),\ \ [r_{ij}] := \begin{cases} 0 & , \quad i = j \\ -[m_{ij}] & , \quad i \neq j \end{cases}$$

$$[D^{-1}] := \text{diag}\left(\frac{1}{[m_{ij}]}\right)$$

$$\begin{cases} [x]^\circ = [\underline{x}^\circ, \overline{x}^\circ] \in \mathbb{IR}^n, \ \ \underline{x}^\circ \geq 0 \\ [x]^{k+1} = \max\left\{0, [D^{-1}]([R][x]^k - [q])\right\}, k = 0, 1, 2, \ldots (T) \end{cases}$$

a) $\lim_{k\to\infty}[x]^k = [x*]$

$$[x^*] = \max\left\{0, [D^{-1}]([R][x^*] - [q])\right\},\ \ [x^*] \text{ unique;}$$

b) $\Sigma \subseteq [x^*]$

$M = (m_{ij}) \in \mathbb{R}^{n\times n}$

Comparison matrix: $<M> = (<m_{ij}>), <m_{ij}> = \begin{cases} |m_{ij}| & , \quad i = j \\ -|m_{ij}| & , \quad i \neq j \end{cases}$

$M$ is an $H$(adamard)-matrix: $<M>$ is an $M$(inkowski)-matrix

($<m_{ij}> \le 0, i \neq j$ and $<M>^{-1} \ge 0$)

$[M] = ([m_{ij}]) \in \mathbb{IR}^{n\times n}$ is an $H$(Interval)-matrix:

Every $M \in [M]$ is an $H$-matrix

Remark: $[M] = ([m_{ij}])$ and $[m_{ij}] = [\underline{m}_{ij}, \overline{m}_{ij}]$

Let $<[M]> = <[m_{ij}]>$ with $<[m_{ij}]> = \begin{cases} min\left\{|m_{ij}| \;\middle|\; m_{ij} \in [m_{ij}]\right\} & , \quad i = j \\ -|[m_{ij}]| & , \quad i \neq j \end{cases}$

Lemma:

$[M]$ is an $H$-matrix $\iff$ $<[M]>$ is an $M$-matrix

Let $[a] = [\underline{a}, \overline{a}]$. Then

$$\max\{0, [a]\} := \begin{cases} 0 & , \ \overline{a} \leq 0 \\ [a] & , \ \underline{a} \geq 0 \\ [0, \overline{a}] & , \ \underline{a} < 0 < \overline{a} \end{cases}$$

$[a] = ([a_i]) \in \mathbb{IR}^n$. Then

$$\max\{0, [a]\} := (\ \max\{0, [a_i]\})$$

Properties: $[a], [b] \in I\mathbb{R}$,

1. $d(\max\{0, [a]\}) \leq d[a]$

2. $q(\max\{0, [a]\}\,,\ \max\{0, [b]\}) \leq q([a], [b])$

   $(q([x], [y]) = \max\left\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\right\}, [x] = [\underline{x}, \overline{x}], [y] = [\underline{y}, \overline{y}])$

3. $[a] \subseteq [b] \Rightarrow \max\ \{0, [a]\} \subseteq \max\ \{0, [b]\}$

Proof of Theorem 1:

a) $g: \mathbb{IR}^n \to \mathbb{IR}^n, \qquad g([x]) = \max\left\{0, [D^{-1}]\,([R][x] - [q])\right\}$

$$q\,(g([x])\,, g([y]) \leq q\left([D^{-1}]([R][x] - [b]), [D^{-1}]([R][y] - [b])\right)$$
$$\leq |[D^{-1}]| \cdot |[R]| q([x], [y]).$$

Let $A := B - C$ where $B = <[D]>, C = |[R]| \Rightarrow$

$<[M]> = B - C$ is an $M$-matrix (by assumption) $\Rightarrow$

$\rho(B^{-1}C) = \rho(<[D]>^{-1} \cdot |[R]|) = \rho(|[D^{-1}]| \cdot |[R]|) < 1 \Rightarrow$

$g$ is a $B^{-1}C-$ contraction $\Rightarrow \exists_1 [x^*] = g([x^*])$;

(T) is convergent to $[x^*]$ for all $[x]^\circ \in \mathbb{IR}^n$.

b) Let $z \in \sum$, that is, there exsits an $M \in [M]$ and a $q \in [q]$ s.t.

$$z \geq 0, \;\; q + Mz \geq 0, \;\; z^T(q + Mz) = 0.$$

Let $D = \text{diag } (m_{ij})$, $R = D - M$. Then

$$0 = \min \{z, q + Mz\} = \min \left\{z, D \; (D^{-1}q + D^{-1}Mz)\right\}.$$

Since $D \geq 0$, this is equivalent to

$$0 = \min \left\{z, D^{-1}q + D^{-1}(D - R)z\right\}$$

$$= \min \left\{z, D^{-1}q + z - D^{-1}Rz\right\}$$

$$= z + \min \left\{0, D^{-1}(q - Rz)\right\} \Rightarrow$$

$$z = -\ \min \left\{0, D^{-1}(q - Rz)\right\}$$

$$= \max \left\{0, D^{-1}(Rz - q)\right\} \in \max \left\{0, [D^{-1}]([R]z - [q])\right\} =: [x]^1$$

$$[x]^\circ := z \ \Rightarrow z \in [x]^k,\ k \geq 0 \ \Rightarrow z \in [x^*].$$

Theorem 2: $[M] = [D] - [L] - [U], [M]$ $H$-matrix, $\underline{m}_{ij} > 0, i = 1, 2, \ldots, n.$

$$0 < \omega < \frac{2}{1+\rho(|[D^{-1}]| \cdot |[R]|)} \qquad ([R] = [L] + [U])$$

$$[x]^\circ = [\underline{x}^\circ, \overline{x}^\circ] \in \mathbb{IR}^n, \underline{x}^\circ \geq 0$$

$$[x]^{k+1} = \max\left\{0, (1-\omega)[x]^k + \omega[D^{-1}]([L][x]^{k+1} + [U][x]^k - [q]\right\},$$

$$k = 0, 1, 2, \ldots (SOR)$$

a) $\lim_{k\to\infty}[x]^k = [x^*]$,

$$[x]^* = \max\left\{0, (1-\omega)[x^*] + \omega[D^{-1}]([L][x^*] + [U][x^*] - [q])\right\}$$

b) $\sum \subseteq [x]^*$.

Theorem 3: Assumptions as in Theorems 1 and 2

$$[x^*] = \max\left\{0, [D^{-1}]([R][x^*] - [q])\right\}$$

$$[x^{**}] = \max\left\{0, (1-\omega)[x^{**}] + \omega[D^{-1}]([R][x^{**}] - [q]\right\}$$

$$0 < \omega < \frac{2}{1+\rho(|[D^{-1}]|\cdot|[R]|)}$$

$\Rightarrow \quad [x^*] \subseteq [x^{**}]$

$0 < \omega < 1: \quad [x^*] = [x^{**}]$

Symmetric Single Step Method:

$$[x]^{k+\frac{1}{2}} = \max\left\{0, [D^{-1}]\left([L][x]^{k+\frac{1}{2}} + [U][x]^{k} - [q]\right)\right.$$

$$[x]^{k+1} = \max\left\{0, [D^{-1}]\left([L][x]^{k+\frac{1}{2}} + U[x]^{k+1} - [q]\right)\right.$$

$k = 0, 1, 2, \ldots$ .

Under the assumptions of Theorem 1:

$$\lim_{k\to\infty}[x]^{k} = [x^*], \quad [x^*] = \max\left\{0, [D^{-1}]([R][x^*] - [q]\right\}$$

Similar for (SSOR)

$$[x^*] \subseteq [x]^\circ$$

$$[x]^{k+1} = \max \left\{0, [D^{-1}] \left([R][x]^k - [q]\right)\right\} \cap [x]^k \ , \ k = 0, 1, 2, \ldots \ .$$

(TI) , Total step method with intersection

For $i = 1, 2, \ldots, n$ :

$$[x_i]^{k+1} = \max\left\{0, \frac{1}{[m_{ij}]} \left(-\sum_{j=1}^{i-1} [m_{ij}][x_j]^{k+1} - \sum_{j=i+1}^{n} [m_{ij}][x_j]^k - [q_i]\right)\right\} \cap [x_i]^k$$

$$k = 0, 1, 2, \ldots \ .$$

(SIC), Single step method with intersection componentwise

For $i = 1, 2, \ldots, n$ :

$$[x_i]^{k+\frac{1}{2}} = \max\left\{0, \frac{1}{[m_{ii}]}\left(-\sum_{j=1}^{i-1}[m_{ij}][x_j]^{k+\frac{1}{2}} - \sum_{j=i+1}^{n}[m_{ij}][x_j]^{k} - [q_i]\right)\right\} \cap [x_i]^{k}$$

For $i = n, n-1, \ldots, 1$ :

$$[x_i]^{k+1} = \max\left\{0, \frac{1}{[m_{ii}]}\left(-\sum_{j=1}^{i-1}[m_{ij}][x_j]^{k+\frac{1}{2}} - \sum_{j=i+1}^{n}[m_{ij}][x_j]^{k+1} - [q_i]\right)\right\} \cap [x_i]^{k+}$$

$k = 0, 1, 2, \ldots$ .

(SSIC), Symmetric single step method with intersection component-wise

Theorem 4:

$$(TI) \quad \geq \quad (SIC) \quad \geq \quad (SSIC)$$

Theorem 1 (Supplement)

Let $[M] = ([m_{ij}])$ be an $M$-matrix.

Then $[x^*] = \text{hull }(\sum)$.

Alefeld, G., Schäfer, U.: Iterative Methods for linear Complementarity Problems with Interval Data. Computing 70, 235-259 (2003)