General Interval FEM Program Based on Sensitivity Analysis

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Abstract. Today there are many methods for solution of equation with interval parameters (Moens and Vandepitte, 2005). Unfortunately there are very few efficient methods which can be directly applied for solution of complex engineering problems. Sensitivity analysis method (Pownuk, 2004) gives very good inner approximation of the exact solution set. This method was implemented in C++ language by the author and the program can be recompiled on Windows, Linux and Solaris operating systems. The program is able to solve 1D, 2D and 3D linear problems of electrostatics with interval parameters.

Additionally it is possible to solve problems with uncertain functional parameters (Pownuk, 2006). In order to do that it is necessary to create special finite elements. It is possible to consider also uncertain shapes The program is very universal and can be applied to the solution of complex engineering problem. The program is a part web application, which is written in php language and can be run on the web page http://andrzej.pownuk.com.

Keywords: interval stresses, stress distribution, sensitivity analysis, functional parameters

1. Design of structures with the interval parameters

One of the simplest method of modelling of uncertain parameters is based on the intervals (Moore, 1966). In that case in order to describe values of the parameter p it is necessary to know only two numbers i.e. upper $\bar{p}$ and lower bound $\underline{p}$.

In civil and mechanical engineering one of the most popular method of mathematical modeling of engineering structures is the finite element method (Zienkiewicz and Taylor, 2000). The FEM method leads to the following system of parameter dependent system of linear or nonlinear equations

$$K(p)u(p) = Q(p) \tag{1}$$

where K is the stiffness matrix, Q is the right hand side and p is the vector of uncertain parameters

$$p = [p_1, ..., p_m]^T. \tag{2}$$

In this paper the following notation for the interval parameters and the interval functions will be applied. If we have the function $f(p)$ then

$$f(\mathbf{p}) = \{f(p) : p \in \mathbf{p}\} \tag{3}$$

$$\mathbf{f}(\mathbf{p}) = \Box f(\mathbf{p}) = \Box\{f(p) : p \in \mathbf{p}\} \tag{4}$$

where $\mathbf{p}$ is the interval or a vector of the interval parameters. Function f can be real valued or vector valued. $\mathbf{p}$ can be an interval in R (i.e. $\mathbf{p} = [\underline{p}_i, \overline{p}_i] \subset R$) or in R^m (i.e. $\mathbf{p} = [\underline{p}_1, \overline{p}_1] \times ... \times [\underline{p}_m, \overline{p}_m]$). If the parameters p_i belong to some know intervals $p_i \in [\underline{p}_i, \overline{p}_i]$, then the solution can be defined as the smallest interval which contain the exact solution set.

$$u(\mathbf{p}) = \{u : K(p)u(p) = Q(p), p \in \mathbf{p}\} \tag{5}$$

$$\mathbf{u}(\mathbf{p}) = \square u(\mathbf{p}) = \square\{u : K(p)u(p) = Q(p), p \in \mathbf{p}\} \tag{6}$$

2. Sensitivity analysis method

There are different methods of calculation of the set (6) (Moens and Vandepitte, 2005; Neumaier, 1990). One of the simplest and most efficient method of solution of system of equations with the interval parameters is the sensitivity analysis method (Pownuk, 2004).

Sensitivity analysis method for general explicit function $u_i = u_i(p)$.

1. Calculate the mid point solution $u(p_0)$ from the following system of equations

$$u_0 = u(p_0) \tag{7}$$

where $p_0 = \text{mid}(\mathbf{p})$.

2. Calculate the sensitivity $\frac{\partial u(p_0)}{\partial p_i}$ at the mid point p_0.

3. Find the combination of parameters which corresponds to the extreme values of the solution.

$$If \ \frac{\partial u_i(p_0)}{\partial p_j} \geqslant 0 \ then \ p_{i,j}^{max} = \overline{p}_j, p_{i,j}^{min} = \underline{p}_j, \tag{8}$$

$$if \ \frac{\partial u_i(p_0)}{\partial p_j} < 0 \ then \ p_{i,j}^{max} = \underline{p}_j, p_{i,j}^{min} = \overline{p}_j. \tag{9}$$

Combination of endpoints which correspond to the extreme value of function $u_i = u_i(p)$ will be denoted in the following way

$$p_i^{min} = (p_{i,1}^{min}, p_{i,2}^{min}, ..., p_{i,m}^{min}), \tag{10}$$

$$p_i^{max} = (p_{i,1}^{max}, p_{i,2}^{max}, ..., p_{i,m}^{max}). \tag{11}$$

4. Create a list L of all critical endpoints combinations.

$$L = \{p_1^{min}, p_1^{max}, p_2^{min}, p_2^{max}, ..., p_m^{min}, p_m^{max}\} = \tag{12}$$

$$= \{p_1, p_2, ..., p_{2m}\} \tag{13}$$

5. Now it is possible to create a new list L^*, which contain only different endpoints

$$L^* = \{p_1^*, p_1^*, ..., p_{n_*}^*\}. \tag{14}$$

6. For all elements in the list L^* calculate a value of the vector u

$$u_{i,j}^* = u_i(p_j^*), \quad for \ j = 1, ..., n^*. \tag{15}$$

7. Calculate the extreme values of the solution

$$\underline{u}_i = \min\{u_i(p_0), u_{i,1}^*, u_{i,2}^*, ..., u_{i,n^*}^*\}, \quad \overline{u}_i = \max\{u_i(p_0), u_{i,1}^*, u_{i,2}^*, ..., u_{i,n^*}^*\}. \tag{16}$$

The results are exact if the sign of the derivative is constant.

3. Interval functional parameters

3.1. Equations with interval functional parameters

In order to get reliable results it is possible to approximate the values of the unknown function p by using some upper and lower bounds

$$p(x) \in [\underline{p}, \overline{p}] = \mathbf{p} \tag{17}$$

Better approximation can be obtained using functional intervals

$$p(x) \in [\underline{p}(x), \overline{p}(x)] = \mathbf{p}(x) \tag{18}$$

Lets assume that the behaviour of the structure with interval parameters is described by the following equation

$$F(x, u, p) = 0 \tag{19}$$

where u is a vector of the solutions and p is a vector of parameters. The solution of the equation (19) can be defined in the following way (Neumaier, 1990)

$$u(x, \mathbf{p}) = \{u : F(x, u, p) = 0, p(x) \in \mathbf{p}(x)\}, \quad x \in \Omega. \tag{20}$$

The set $u(x, \mathbf{p})$ is in general very complicated (Neumaier, 1990), because of that in applications it is easier to use the smallest interval which contain the exact solution set.

$$\mathbf{u}(x, \mathbf{p}) = \Box u(x, \mathbf{p}) = \Box\{u : F(x, u, p) = 0, p(x) \in \mathbf{p}(x)\}, \quad x \in \Omega. \tag{21}$$

If the equation is not directly dependent on x then the solution set is the following

$$u(\mathbf{p}) = \{u : F(u, p) = 0, p(x) \in \mathbf{p}(x)\}, \tag{22}$$

$$\mathbf{u}(\mathbf{p}) = \Box u(\mathbf{p}) = \Box\{u : F(u, p) = 0, p(x) \in \mathbf{p}(x)\}. \tag{23}$$

3.2. General concept of monotonicity

A map $T : X \to Y$ is monotone if $(X, \geqslant)$ is a partially ordered set and $x, y \in X$, $x \geqslant y \Rightarrow T(x) \geqslant T(y)$. Typically, X will be a subset of a Banach space Y with a cone Y_+ of positive elements and $x \leqslant y$ is equivalent to $y - x \in Y_+$ (Hirsch and Smith, 2005).

3.3. Solution of the equations with the interval functional parameters - general case

In general it is very hard to get the solution set (23) or (21). Fortunately in many applications it is possible to apply the method which is based on sensitivity analysis, Taylor expansion and/or functional derivative (Pownuk, 2006). These methods allow us to get very actuate solution and have low computational complexity.

Let us consider a function $u = u(p)$ where $p : R^n \supset \Omega \to p(x) \in R$, X is a functional space which contain the functions p, u is the function form the space X to the space R i.e. $u : X \ni p \to u(p) \in R$. Lets consider only positive variation of the function p i.e.

$$\delta p(x) = p_1(x) - p_0(x) > 0 \tag{24}$$

where $p_1, p_2 \in X$. If one add positive variation to the function p_0 then the results (i.e. $p_0 + \delta p$) is bigger than the function p_0 i.e.

$$p_0 + \delta p(x) > p_0(x) \tag{25}$$

If the difference $u(p + \delta p) - p(p_0)$ has constant sign the the function u is monotone.

If the function u is differentiable then finite increment of the functions u can be approximated by the differential

$$u(p_0 + \delta p) - u(p_0) = \delta u(p_0, \delta p) + R(p_0, \delta p) \tag{26}$$

where

$$\lim_{\|\delta p\| \to 0} \frac{|R(p_0, \delta p)|}{\|\delta p\|} = 0, \tag{27}$$

and for small variations δp we can write

$$u(p_0 + \delta p) - u(p_0) \approx \delta u(p_0, \delta p) \tag{28}$$

If the differential $\delta u(p_0, \delta p)$ is positive then the function $u = u(p)$ is monotone around the point p_0 (Hirsch and Smith, 2005).

Theorem 1
If the function $u : X \to R$ is differentiable and $\delta u(p_0, \delta p) \geqslant 0$ for all $p \in [\underline{p}, \overline{p}] \subset X$ and some δp, then $u = u(p)$ is monotone in the interval $[\underline{p}, \overline{p}]$.

Proof

$$u(p_0 + \delta p) - u(p_0) = \int\limits_0^1 \delta u(p_0 + t\delta p, \delta p) dt \tag{29}$$

if $\delta u(p_0 + t\delta p, \delta p) \geqslant 0$ then $\int\limits_0^1 \delta u(p_0 + t\delta p, \delta p)dt \geqslant 0$ and then

$$u(p_0 + \delta p) \geqslant u(p_0) \tag{30}$$

i.e. the function u is monotone. Now it is possible to calculate extreme values of the function $u = u(p)$ for $p \in \mathbf{p}$ if the sign of the differential is constant.

General sensitivity analysis with functional parameters

1. **if** $\delta u(p, \delta p) \geqslant 0$ **then** $p^{min} = \underline{p}$, $p^{max} = \overline{p}$.

2. **if** $\delta u(p, \delta p) < 0$ **then** $p^{min} = \overline{p}$, $p^{max} = \underline{p}$.

3. $\underline{u} = u(p^{min})$, $\overline{u} = u(p^{max})$.

The algorithm is not very practical because in general it is hard to verify the sign of the differential $\delta u(p_0, \delta p)$. In order to make that method a little more practical it is necessary to consider some special cases.

3.4. EXTREME VALUES OF THE INTEGRAL IN THE FORM $u(p) = \int\limits_\Omega L(x, p(x))dx$

Differential of the function $u(p) = \int\limits_\Omega L(x, p(x))dx$ has the following form

$$\delta u(p_0, \delta p) = \int\limits_\Omega \frac{\partial L(x, p(x))}{\partial p(x)}\delta p(x)dx = \left\langle \frac{\delta u}{\delta p}, \delta p \right\rangle \tag{31}$$

where

$$\frac{\delta u}{\delta p(x)} = \frac{\partial L(x, p(x))}{\partial p(x)} \tag{32}$$

is the functional derivative of the function $u = u(p)$ and $\langle ., . \rangle$ is the scalar product.

Theorem 2
If $\frac{\delta u}{\delta p(x)} \geqslant 0$ for $p \in [\underline{p}, \overline{p}] \subset X$, then the function $u = u(p)$ is monotone in the interval $\mathbf{p}$.
Proof
If $\frac{\delta u}{\delta p(x)} \geqslant 0$ and $\delta p(x) \geqslant 0$ then $\delta u(p_0, \delta p) = \left\langle \frac{\delta u}{\delta p}, \delta p \right\rangle \geqslant 0$ and according to the theorem 1 the function $u = u(p)$ is monotone.

Now it is possible to use more efficient version of the algorithm

Sensitivity analysis based on functional derivative

1. **if** $\frac{\delta u}{\delta p(x)} \geqslant 0$ **then** $p^{min} = \underline{p}$, $p^{max} = \overline{p}$.
   if $\frac{\delta u}{\delta p(x)} < 0$ **then** $p^{min} = \overline{p}$, $p^{max} = \underline{p}$.
   $\underline{u} = u(p^{min})$, $\overline{u} = u(p^{max})$.

If the sign of the functional derivative is not constant, then it is possible to apply approximate method for finding extreme values of the solutions. According to the equation (28) the finite increment of the functions can be approximated by the differential. If the differential is positive (i.e. $\delta u(p_0, \delta p) \geqslant 0$) then for very small variations δp we can assume that $u(p + \delta p) \geqslant u(p)$. The product $\frac{\delta u}{\delta p(x)} \delta p(x)$ is nonnegative if $\frac{\delta u}{\delta p(x)} \geqslant 0$ and $\delta p(x) \geqslant 0$ or $\frac{\delta u}{\delta p(x)} \leqslant 0$ and $\delta p(x) \leqslant 0$. If we have the function $p_0 \in [\underline{p}, \overline{p}]$ and the value of functional derivative $\frac{\delta u(p_0)}{\delta p(x)}$ is not constant, then it is possible to change the sign of the variation δp is such a way which make the differential positive. It is possible to define the small variations in the following way

$$\delta p^u(x) = \lambda(x) \frac{\delta u(p_0)}{\delta p(x)}, \quad \delta p^l(x) = -\lambda(x) \frac{\delta u(p_0)}{\delta p(x)} \tag{33}$$

where $\lambda(x)$ is an arbitrary positive function. If the variations $\delta p^l, \delta p^u$ are small enough then $\delta u(p_0, \delta p^u) \geqslant 0$, $\delta u(p_0, \delta p^l) \leqslant 0$ and according to the relation (28) we can write

$$u(p_0 + \delta p^u) \geqslant u(p_0) \tag{34}$$

$$u(p_0 + \delta p^l) \leqslant u(p_0) \tag{35}$$

Above described properties can be applied to the creation of approximate algorithm for finding upper and lower bound of the function $u = u(p)$.

Calculation of upper bound $\overline{u}$

1. $p(x) = p_0(x)$

2. choose the function $\lambda(x)$

3. $\delta p^u(x) = \lambda(x) \frac{\delta u(p)}{\delta p(x)}$

4. $p_{old}(x) = p(x)$

5. $p(x) := p(x) + \delta p^u(x)$

6. **if** $p(x) > \overline{p}(x)$ **then** $p(x) = \overline{p}(x)$

7. **if** $p(x) < \underline{p}(x)$ **then** $p(x = \underline{p}(x)$

8. **if** $\|p_{old} - p\| > \varepsilon$ **then** goto step 2

9. $\overline{u} = u(p)$

10. **stop**

The lower bound can be calculated in the similar way.

3.5. EXTREME VALUES OF THE FUNCTIONS AND THE INTEGRALS

In more complicated cases the function $u = u(p)$ is a superposition of algebraic function f and the integrals in the form $\int_{\Omega} L_i(x, p(x))dx$

$$u(p) = f(y_1, ..., y_q) = f(y)$$
$$y_1 = I_1(p) = \int_{\Omega} L_1(x, p(x))dx, ..., y_q = I_q(p) = \int_{\Omega} L_q(x, p(x))dx \tag{36}$$

Differential in this case is equal to:

$$\delta u(p, \delta p) = \sum_i \frac{\partial f(y)}{\partial y_i} I_i(p, \delta p) = \sum_i \frac{\partial f(y)}{\partial y_i} \int_{\Omega} \frac{\partial L_i}{\partial p(x)} \delta p(x) dx \tag{37}$$

Functional derivative can be defined in this case in the following way

$$\frac{\delta u}{\delta p(x)} = \sum_i \frac{\partial f(y)}{\partial y_i} \frac{\delta I_i(p)}{\delta p(x)} = \sum_i \frac{\partial f(y)}{\partial y_i} \frac{\partial L_i(x, p(x))}{\partial p(x)} \tag{38}$$

In matrix notation

$$\frac{\delta u(p)}{\delta p} = \left[\frac{\partial f(y)}{\partial y_1}, ..., \frac{\partial f(y)}{\partial y_p} \right] \begin{bmatrix} \frac{\partial L_1}{\partial p(x)} \\ ... \\ \frac{\partial L_p}{\partial p(x)} \end{bmatrix} \tag{39}$$

If the sign of the functional derivative is constant, then the sign of the differential is constant (for very small perturbations δp) and according to the theorem ?? the function $u = u(p)$ is monotone. In order to calculate the extreme values of the solutions by using the algorithm 3.4. If the sign of the derivative is not constant then it is possible to apply algorithm ?? and ??.

It is also interesting to study the function u in the case when it depend on many functions p_i i.e.

$$u(p) = f(y_1, ..., y_q) = f(y)$$
$$y_1 = I_1(p) = \int_{\Omega} L_1(x, p(x))dx, ..., y_q = I_q(p) = \int_{\Omega} L_q(x, p(x))dx \tag{40}$$

where $p = (p_1, ..., p_m)$. The differential is equal to

$$\delta u(p, \delta p) = \sum_i \frac{\partial f(y)}{\partial y_i} I_i(p, \delta p) = \sum_i \frac{\partial f(y)}{\partial y_i} \int_{\Omega} \left(\sum_j \frac{\partial L_i}{\partial p_j(x)} \delta p_j(x) \right) dx \tag{41}$$

Now it is possible to calculate the functional derivative which is in this case a vector with the following components

$$\frac{\delta u(p)}{\delta p} = \left[\sum_i \frac{\partial f(y)}{\partial y_i} \frac{\partial L_i}{\partial p_1(x)}, ..., \sum_i \frac{\partial f(y)}{\partial y_i} \frac{\partial L_i}{\partial p_m(x)} \right] \tag{42}$$

In matrix notation

$$\frac{\delta u(p)}{\delta p} = \left[\frac{\partial f(y)}{\partial y_1}, ..., \frac{\partial f(y)}{\partial y_p} \right] \left[\begin{array}{ccc} \frac{\partial L_1}{\partial p_1(x)} & \cdots & \frac{\partial L_1}{\partial p_m(x)} \\ \cdots & \cdots & \cdots \\ \frac{\partial L_p}{\partial p_1(x)} & \cdots & \frac{\partial L_p}{\partial p_m(x)} \end{array} \right] \tag{43}$$

The differential is positive if the variations δp_j have the same sign as $\sum_i \frac{\partial f(y)}{\partial y_i} \frac{\partial L_i}{\partial p_j(x)}$. It is also possible to create discrete version of these methods.

4. Sensitivity with respect to changes of the region of integration

4.1. INTRODUCTION

Lets consider a function $u = u(\Omega)$ where Ω is a domain of integration.

$$u(\Omega) = \int_\Omega L(x) dx \tag{44}$$

Lets consider the following increment

$$u(\Omega + \Delta\Omega) - u(\Omega) = \int_{\Omega+\Delta\Omega} L(x)dx - \int_\Omega L(x)dx = \int_{\Delta\Omega} L(x)dx \tag{45}$$

The operation "$\Omega + \Delta\Omega$" is a sum of two set i.e. "$\Omega \cup \Delta\Omega$". If the set is convex then from main value theorem

$$\int_{\Delta\Omega} L(x)dx = |\Delta\Omega| L(x^*) \tag{46}$$

where $x^* \in \Delta\Omega$.

$$\frac{u(\Omega + \Delta\Omega) - u(\Omega)}{|\Delta\Omega|} = L(x^*) \tag{47}$$

In the limit case

$$\frac{\delta u}{\delta\Omega(x)} = \lim_{|\Delta\Omega(x)| \to 0} \frac{u(\Omega + \Delta\Omega(x)) - u(\Omega)}{|\Delta\Omega(x)|} = L(x). \tag{48}$$

If $\underline{\Omega} \subset \Omega \subset \overline{\Omega}$ then extreme values of the function $u = u(p)$ by using sensitivity analysis method.

The inclusion $\subset$ can be treat as the partial order relation $\geqslant$. Because of that it is possible to take into account "set intervals"

$$[\underline{\Omega}, \overline{\Omega}] = \{\Omega : \underline{\Omega} \subset \Omega \subset \overline{\Omega}\}. \tag{49}$$

If the sign of the derivative $\frac{\delta u}{\delta\Omega(x)}$ is not constant then it is possible to create the sets Ω^{max} and Ω^{min} in the following way.

$$\Omega^{max} = \underline{\Omega} \cup \left\{ x : \frac{\delta u}{\delta\Omega(x)} \geqslant 0, x \in \overline{\Omega} - \underline{\Omega} \right\} \tag{50}$$

$$\Omega^{min} = \underline{\Omega} \cup \left\{ x : \frac{\delta u}{\delta \Omega(x)} < 0, x \in \overline{\Omega} - \underline{\Omega} \right\} \tag{51}$$

Extreme values of the function $u = u(\Omega)$ are equal $\underline{u} = u(\Omega^{min})$, $\overline{u} = u(\Omega^{max})$.
The function $u = u(\Omega)$ may be a superposition of algebraic function and the integral.

$$u(\Omega) = f(y), \quad y = \int_{\Omega} L(x)dx \tag{52}$$

$$\frac{\delta u}{\delta \Omega(x)} = \frac{df(p)}{dy} \frac{\delta}{\delta \Omega(x)} \int_{\Omega} L(x)dx = \frac{df(p)}{dy} L(x) \tag{53}$$

The function u can be dependent on many integrals.

$$u(\Omega) = f(y), \quad y_1 = \int_{\Omega} L_1(x)dx, ..., y_n = \int_{\Omega} L_p(x)dx \tag{54}$$

$$\frac{\delta u}{\delta \Omega(x)} = \sum_i \frac{df(p)}{dy_i} \frac{\delta}{\delta \Omega(x)} \int_{\Omega} L_i(x)dx = \sum_i \frac{df(p)}{dy_i} L_i(x) \tag{55}$$

4.2. Moment of inertia of cross-section with uncertain shape

Polar moment of inertia

$$I_0(\Omega) = \int_{\Omega} r^2 d\Omega = \iint_{\Omega} (x^2 + y^2)dxdy \tag{56}$$

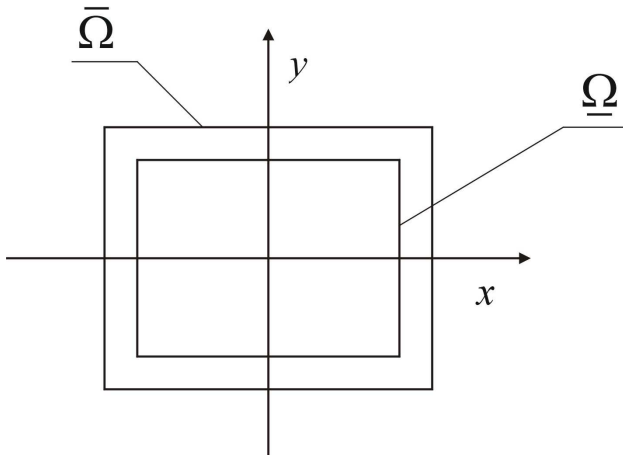Because the limit is positive in the set $\overline{\Omega} - \underline{\Omega}$

Figure 1. Uncertain shape of cross-section

Andrzej Pownuk

$$\frac{\delta I_0}{\delta \Omega(x,y)} = x^2 + y^2 \geqslant 0 \tag{57}$$

then

$$\underline{I}_0 = I_0(\underline{\Omega}) = \iint\limits_{\underline{\Omega}} (x^2 + y^2)dxdy, \quad \overline{I}_0 = I_0(\overline{\Omega}) = \iint\limits_{\overline{\Omega}} (x^2 + y^2)dxdy \tag{58}$$

In the case of product moment of inertia

$$I_{xy}(\Omega) = \iint\limits_{\Omega} xydxdy \tag{59}$$

the limit

$$\frac{\delta I_{xy}}{\delta \Omega(x,y)} = xy \tag{60}$$

is sometimes positive and sometimes negative. From the picture 2 we can see that $xy \geqslant 0$ in the
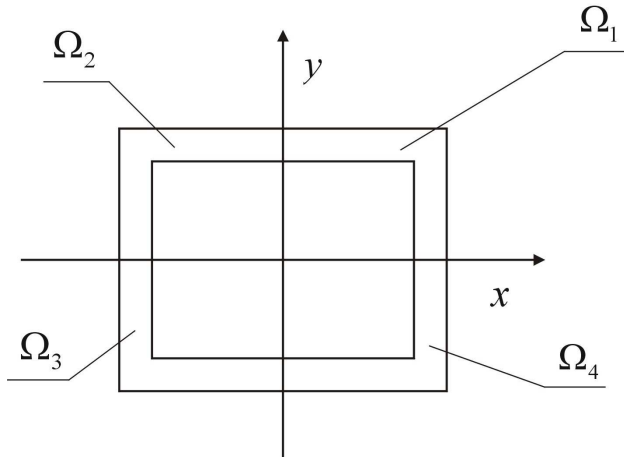
Figure 2. Uncertain shape

sets Ω_1 and Ω_3. $xy \leqslant 0$ in the set Ω_2 and Ω_4. Because of that

$$\underline{I}_{xy} = I_{xy}(\underline{\Omega} \cup \Omega_2 \cup \Omega_4) = \iint\limits_{\underline{\Omega} \cup \Omega_2 \cup \Omega_4} (x^2 + y^2)dxdy \tag{61}$$

$$\overline{I}_{xy} = I_{xy}(\underline{\Omega} \cup \Omega_1 \cup \Omega_3) = \iint\limits_{\underline{\Omega} \cup \Omega_1 \cup \Omega_3} (x^2 + y^2)dxdy \tag{62}$$

5. General case

In general it is possible to consider a functional which is dependent on parameters $h_i \in \mathbf{p}$, functional parameters $p_i(x) \in \mathbf{p}_i(x)$ and integrals which are dependent on the sets Ω_i

$$u = F(h_1, ..., h_m, y_1, ..., y_p, z_1, ..., z_q) \tag{63}$$

$$y_1 = \int_{\Gamma_1} L_1(x, p_1(x), ..., p_k(x)) dx \tag{64}$$

$$y_2 = \int_{\Gamma_2} L_2(x, p_1(x), ..., p_k(x)) dx \tag{65}$$

$$... \tag{66}$$

$$y_q = \int_{\Gamma_q} L_q(x, p_1(x), ..., p_k(x)) dx \tag{67}$$

$$z_1 = \int_{\Omega_1} \Psi_1(x, p_1(x), ..., p_k(x)) dx \tag{68}$$

$$z_2 = \int_{\Omega_2} \Psi_2(x, p_1(x), ..., p_k(x)) dx \tag{69}$$

$$... \tag{70}$$

$$z_q = \int_{\Omega_q} \Psi_q(x, p_1(x), ..., p_k(x)) dx \tag{71}$$

If the sign of each derivative is constant then it is possible to apply sensitivity analysis to each uncertain parameters separately.

6. Direct method of calculation of sensitivity from differential equation

6.1. Sensitivity with respect to real valued parameters

Lets us consider tension-compression differential equation

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + n = 0 \tag{72}$$

with the following boundary condition

$$EA\frac{du(0)}{dx} = P, \quad u(0) = 0 \tag{73}$$

After integration we will get

$$EA\frac{du}{dx} + \int\limits_0^x ndx = EA\frac{du(0)}{dx} \tag{74}$$

$$EA\frac{du}{dx} + \int\limits_0^x ndx = P \tag{75}$$

$$\frac{du}{dx} = \frac{P}{EA} - \frac{1}{EA}\int\limits_0^x ndx \tag{76}$$

$$u(x) = u(0) + \int\limits_0^x \frac{P}{EA}d\eta - \int\limits_0^x \left(\frac{1}{EA}\int\limits_0^\eta nd\xi\right)d\eta \tag{77}$$

$$u(x) = \int\limits_0^x \frac{P}{EA}d\eta - \int\limits_0^x \left(\frac{1}{EA}\int\limits_0^\eta nd\xi\right)d\eta \tag{78}$$

For constant values of E, A and n we will get

$$u(x) = \frac{Px}{EA} - \frac{nx^2}{2EA} \tag{79}$$

Partial derivative of the solution is equal to

$$\frac{\partial u(x)}{\partial E(y)} = -\frac{Px}{E^2 A} + \frac{nx^2}{2E^2 A} \tag{80}$$

Functional derivative of the differential equation with respect to the uncertain parameter p_i

$$\frac{\partial}{\partial p_i}\left[\frac{d}{dx}\left(EA\frac{du}{dx}\right) + n\right] = 0 \tag{81}$$

$$\frac{d}{dx}\left(\frac{\partial(EA)}{\partial p_i}\frac{du}{dx}\right) + \frac{d}{dx}\left(EA\frac{d}{dx}\left(\frac{\partial u}{\partial p_i}\right)\right) + \frac{\partial n}{\partial p_i} = 0 \tag{82}$$

For example if $p_i = E$ then

$$\frac{d}{dx}\left(\frac{\partial(EA)}{\partial E}\frac{du}{dx}\right) + \frac{d}{dx}\left(EA\frac{d}{dx}\left(\frac{\partial u}{\partial E}\right)\right) + \frac{\partial n}{\partial E} = 0 \tag{83}$$

$$\frac{d}{dx}\left(A\frac{du}{dx}\right) + \frac{d}{dx}\left(EA\frac{d}{dx}\left(\frac{\partial u}{\partial E}\right)\right) = 0 \tag{84}$$

After integration

$$A\frac{du}{dx} + EA\frac{d}{dx}\left(\frac{\partial u}{\partial E}\right) = C \tag{85}$$

Derivative of boundary conditions

$$\frac{d}{dx}\left(\frac{\partial u(0)}{\partial E}\right) = -\frac{P}{E^2 A}, \quad \frac{\partial u(0)}{\partial E} = 0 \tag{86}$$

then

$$\frac{P}{E} - EA\frac{P}{E^2 A} = C \Rightarrow 0 = C \tag{87}$$

From boundary conditions we will get

$$A\frac{du}{dx} + EA\frac{d}{dx}\left(\frac{\partial u}{\partial E}\right) = 0 \tag{88}$$

$$\frac{d}{dx}\left(\frac{\partial u}{\partial E}\right) = -\frac{1}{E}\frac{du}{dx} \tag{89}$$

$$\frac{d}{dx}\left(\frac{\partial u}{\partial E}\right) = -\frac{1}{E}\left(\frac{P}{EA} - \frac{1}{EA}\int_0^x n d\eta\right) \tag{90}$$

$$\frac{d}{dx}\left(\frac{\partial u}{\partial E}\right) = -\frac{P}{E^2 A} + \frac{1}{E^2 A}\int_0^x n d\eta \tag{91}$$

After integration

$$\frac{\partial u}{\partial E} = \frac{\partial u(0)}{\partial E} - \int_0^x \frac{P}{E^2 A}d\xi + \int_0^x\left(\frac{1}{E^2 A}\int_0^\xi n d\eta\right)d\xi \tag{92}$$

$$\frac{\partial u}{\partial E} = -\int_0^x \frac{P}{E^2 A}d\xi + \int_0^x\left(\frac{1}{E^2 A}\int_0^\xi n d\eta\right)d\xi \tag{93}$$

For constant values

$$\frac{\partial u}{\partial E} = -\frac{Px}{E^2 A} + \frac{nx^2}{2E^2 A} \tag{94}$$

Using this method it is possible to avoid approximation errors.

6.2. SENSITIVITY WITH RESPECT TO FUNCTIONAL PARAMETERS

The solution of the equation (72) is given by the formula (78). The functional derivative with the respect the the values of Young modulus $E(y)$ is equal to

$$\frac{\delta u(x)}{\delta E(y)} = \frac{\delta}{\delta E(y)}\int_0^x \frac{P}{EA}d\eta - \frac{\delta}{\delta E(y)}\int_0^x\left(\frac{1}{EA}\int_0^\eta n d\xi\right)d\eta \tag{95}$$

$$\frac{\delta u(x)}{\delta E(y)} = -\frac{P}{E^2(y)A(y)} + \frac{1}{E^2(y)A(y)} \int_0^y n d\xi \tag{96}$$

It is possible to calculate the functional derivative of the solution of the equation (72) with respect of the functional parameter $E = E(y)$

$$\frac{d}{dx}\left(E(x)A(x)\frac{du(x,E)}{dx}\right) + n(x) = 0 \tag{97}$$

$$\frac{d}{dx}\left((E(x) + \delta E(x))A(x)\frac{du(x,E+\delta E)}{dx}\right) + n(x) = 0 \tag{98}$$

The last equation for a small perturbation can be written in the following way

$$u(x, E + \delta E) \approx u(x, E) + \delta u_E(x, \delta E) \tag{99}$$

After neglecting quadratic terms we will get

$$\begin{aligned} &\frac{d}{dx}\left(E(x)A(x)\frac{du(x,E)}{dx}\right) + \frac{d}{dx}\left(\delta E(x)A(x)\frac{du(x,E)}{dx}\right) + \\ &+ \frac{d}{dx}\left(E(x)A(x)\frac{d}{dx}\delta u_E(x,\delta E)\right) + n(x) = 0 \end{aligned} \tag{100}$$

If we subtract the equations (97) and (100) the result is

$$\frac{d}{dx}\left(\delta E(x)A(x)\frac{du(x,E)}{dx}\right) + \frac{d}{dx}\left(E(x)A(x)\frac{d}{dx}\delta u_E(x,\delta E)\right) = 0 \tag{101}$$

After integration we will get

$$\delta E(x)A(x)\frac{du(x,E)}{dx} + E(x)A(x)\frac{d}{dx}\delta u_E(x,\delta E) = C \tag{102}$$

The functional derivative of the boundary conditions is given by the following formulas

$$u(0, E) = 0, \tag{103}$$

$$u(0, E + \delta E) = 0, \tag{104}$$

$$u(0, E) + \delta u_E(0, \delta E) = 0 \tag{105}$$

then

$$\delta u(0, \delta E) = 0 \tag{106}$$

$$\frac{d}{dx}u(0, E) = \frac{P}{E(0)A(0)}, \tag{107}$$

$$\frac{d}{dx}u(0, E + \delta E) = \frac{P}{(E(0) + \delta E(0))A(0)}, \tag{108}$$

$$\frac{d}{dx}u(0, E) + \frac{d}{dx}\delta u_E(0, \delta E) = \frac{P}{E(0)A(0)} - \frac{P\delta E(0)}{E^2(0)A(0)} \tag{109}$$

then

$$\frac{d}{dx}\delta u_E(0,\delta E) = -\frac{P\delta E(0)}{E^2(0)A(0)} \tag{110}$$

For $x = 0$ we have

$$\delta E(0)A(0)\frac{du(0,E)}{dx} + E(0)A(0)\frac{d}{dx}\delta u_E(0,\delta E) = C \tag{111}$$

From boundary conditions

$$\delta E(0)A(0)\frac{P}{E(0)A(0)} - E(0)A(0)\frac{P\delta E(0)}{E^2(0)A(0)} = C \tag{112}$$

$$0 = C \tag{113}$$

Now the equation has the following form

$$\delta E(x)A(x)\frac{du(x,E)}{dx} + E(x)A(x)\frac{d}{dx}\delta u_E(x,\delta E) = 0 \tag{114}$$

$$\frac{d}{dx}\delta u_E(x,\delta E) = -\frac{\delta E(x)}{E(x)}\frac{du(x,E)}{dx} \tag{115}$$

From the equation (76)

$$\frac{d}{dx}\delta u_E(x,\delta E) = -\frac{\delta E(x)}{E(x)}\left(\frac{P}{E(x)A(x)} - \frac{1}{E(x)A(x)}\int\limits_0^x n(x)dx\right) \tag{116}$$

$$\frac{d}{dx}\delta u_E(x,\delta E) = -\frac{P\delta E(x)}{E^2(x)A(x)} + \frac{\delta E(x)}{E^2(x)A(x)}\int\limits_0^x n(\eta)d\eta \tag{117}$$

After integration

$$\begin{aligned}\delta u_E(x,\delta E) = \delta u_E(0,\delta E) - \int\limits_0^x \frac{P\delta E(\xi)}{E^2(\xi)A(\xi)}d\xi + \\ + \int\limits_0^x \left(\frac{\delta E(\xi)}{E^2(\xi)A(\xi)}\int\limits_0^\xi n(\eta)d\eta\right)d\xi\end{aligned} \tag{118}$$

for $x = 0$ we know that $\delta u(0,\delta E) = 0$, then

$$\delta u_E(x,\delta E) = -\int\limits_0^x \frac{P\delta E(\xi)}{E^2(\xi)A(\xi)}d\xi + \int\limits_0^x \left(\frac{\delta E(\xi)}{E^2(\xi)A(\xi)}\int\limits_0^\xi n(\eta)d\eta\right)d\xi \tag{119}$$

$$\delta u_E(x,\delta E) = \int\limits_0^x \left(\frac{-P}{E^2(\xi)A(\xi)} + \frac{1}{E^2(\xi)A(\xi)}\int\limits_0^\xi n(\eta)d\eta\right)\delta E(\xi)d\xi \tag{120}$$

then

$$\frac{\delta u(x)}{\delta E(\xi)} = \frac{-P}{E^2(\xi)A(\xi)} + \frac{1}{E^2(\xi)A(\xi)}\int\limits_0^\xi n(\eta)d\eta \tag{121}$$

7. FEM with uncertain functional parameters

Finite element method lead to the following parameter dependent system of equations (Zienkiewicz and Taylor, 2000)

$$K(p)u(p) = Q(p) \tag{122}$$

where K is the stiffness matrix, Q is the load vector and u is the vector of the solutions. The functional derivative $\frac{\delta u(p)}{\delta p_i(x)}$ of the solution can be calculated from the following equation

$$K(p)\frac{\delta u(p)}{\delta p_i(x)} = \frac{\delta Q(p)}{\delta p_i(x)} - \frac{\delta K(p)}{\delta p_i(x)}u(p). \tag{123}$$

The solution $\frac{\delta u(p)}{\delta p_i(x)}$ can be applied in the algorithms, which are described in the previous sections. It is not possible to calculate the functional derivative $\frac{\delta u(p)}{\delta p_i(x)}$ in all points $x \in \Omega$. Because of that functional derivative should be calculated in as many grid points as possible x_k. The sign of the functional derivative $\frac{\delta u(p)}{\delta p_i(x)}$ is calculated by using the nearest grid points x_k i.e. $\frac{\delta u(p)}{\delta p_i(x_k)}$.

8. Postprocessing of the interval solution
based on sensitivity analysis

8.1. 3D ELASTICITY

In structural mechanics solution of the system of equations (122) is used for calculations of other mechanical quantities like for example stress and strain. In linear elasticity the relation between the strain tensor ε and displacement vector u is the following

$$\varepsilon(x) = \frac{1}{2}\left(\nabla^T u(x) + \nabla u(x)\right) \tag{124}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{125}$$

In cartesian coordinate system it is possible to approximate the displacement field $u_i(x)$ (i is a direction of the displacement i.e. $x, y, z, \varphi_x, \varphi_y, \varphi_z$) in the element Ω_e using shape functions $N_{ek}(x)$ (e is a number of element, k is a number of node) and the values of the function u_i in the nodal points x_{ek} (usually $u_{eki} = u_i(x_{ek})$, however u_{eki} can be also defined using derivatives of the function u_i, e is a number of element, k is a number of node, i is a direction of the displacement)

$$u_{ei}(x) \approx \sum_k N_{ek}(x)u_{eki} \tag{126}$$

From the equation (124) and (126) we have

$$\varepsilon_{eij} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\left(\sum_k N_{ek}(x)u_{eki}\right) + \frac{\partial}{\partial x_i}\left(\sum_k N_{ek}(x)u_{ekj}\right)\right) = \tag{127}$$

$$= \frac{1}{2} \left(\sum_k \frac{\partial N_{ek}(x)}{\partial x_j} u_{eki} + \sum_k \frac{\partial N_{ek}(x)}{\partial x_i} u_{eki} \right) = \tag{128}$$

$$= \frac{1}{2} \sum_{k,p} \left(\frac{\partial N_{ek}(x)}{\partial x_j} \delta_{pi} + \frac{\partial N_{ek}(x)}{\partial x_i} \delta_{pj} \right) u_{ekp} = \tag{129}$$

$$= \sum_{k,p} B_{eijkp} u_{ekp} \tag{130}$$

then

$$B_{eijkp} = \frac{1}{2} \sum_{k,p} \left(\frac{\partial N_{ek}(x)}{\partial x_j} \delta_{pi} + \frac{\partial N_{ek}(x)}{\partial x_i} \delta_{pj} \right) \tag{131}$$

Relation between the global solution vector u_q (q is a number of degree of freedom in the solution vector) and the vector of local solution of the elements u_{ekp} (e is a number of element, k is a number of node in the element e, p is a direction of the displacement) is the following

$$u_{ekp} = \sum_q U_{ekpq} u_q \tag{132}$$

Sensitivity of the displacements

$$\frac{\partial u_{ekp}}{\partial p_j} = \sum_q U_{ekpq} \frac{\partial u_q}{\partial p_j} \tag{133}$$

In the case of linear elastic materials the relation between the stress σ_{ij} and strain ε_{ij} is the following

$$\sigma_{emn} = \sum_{i,j} C_{emnij} \varepsilon_{eij} = \sum_{i,j,k,p} C_{emnij} B_{eijkp} u_{ekp} \tag{134}$$

The sensitivity of the strain field can be calculated as a derivative

$$\frac{\partial \varepsilon_{eij}}{\partial p_l} = \frac{\partial}{\partial p_l} \left(\sum_{k,p} B_{eijkp} u_{ekp} \right) = \tag{135}$$

$$= \sum_{k,p} \left(\frac{\partial B_{eijkp}}{\partial p_l} u_{ekp} + B_{eijkp} \frac{\partial u_{ekp}}{\partial p_l} \right) \tag{136}$$

or in the case of functional parameters

$$\frac{\delta \varepsilon_{eij}}{\delta p_l(x)} = \frac{\delta}{\delta p_l(x)} \left(\sum_{k,p} B_{eijkp} u_{ekp} \right) = \tag{137}$$

$$= \sum_{k,p} \left(\frac{\delta B_{eijkp}}{\delta p_l(x)} u_{ekp} + B_{eijkp} \frac{\delta u_{ekp}}{\delta p_l(x)} \right) \tag{138}$$

The sensitivity of the stress field can be calculated from the equation (134)

$$\frac{\partial \sigma_{emn}}{\partial p_l} = \frac{\partial}{\partial p_l}\left(\sum_{i,j,k,p} C_{emnij}B_{eijkp}u_{ekp}\right) = \tag{139}$$

$$= \sum_{i,j,k,p}\left(\frac{\partial(C_{emnij}B_{eijkp})}{\partial p_l}u_{ekp} + B_{eijkp}C_{emnij}\frac{\partial u_{ekp}}{\partial p_l}\right) \tag{140}$$

or in the case of functional parameters

$$\frac{\delta \sigma_{emn}}{\delta p_l(x)} = \frac{\delta}{\delta p_l(x)}\left(\sum_{i,j,k,p} C_{emnij}B_{eijkp}u_{ekp}\right) \tag{141}$$

$$= \sum_{i,j,k,p}\left(\frac{\delta(C_{emnij}B_{eijkp})}{\delta p_l(x)}u_{ekp} + B_{eijkp}C_{emnij}\frac{\delta u_{ekp}}{\delta p_l(x)}\right) \tag{142}$$

If we know the derivatives of the strain and stress field then it is possible to calculate the extreme values of the solution using the methods which are described in the previous sections.

Potential energy can be calculated as

$$V = \sum_{e,m,n,i,j}\int_{\Omega} C_{emnij}\varepsilon_{eij}\varepsilon_{emn}d\Omega - \sum_{e,i}\int_{\Omega} f_{ei}u_{ei}d\Omega \tag{143}$$

where f_{ei} are the loads. The local stiffness matrix can be calculated from the following formula

$$K_{ekplq} = \sum_{e,m,n,i,j}\int_{\Omega} C_{emnij}B_{emnkp}B_{eijlp}d\Omega \tag{144}$$

Global stiffness matrix

$$K_{\alpha\beta} = \sum_{e} K_{e\alpha\beta} \tag{145}$$

where

$$K_{e\alpha\beta} = \sum_{k,p,l,q} K_{ekplq}U_{ekp\alpha}U_{ekp\beta} \tag{146}$$

Above relation is linear that is way it is possible to calculate sensitivity of global stiffness matrix using linear relation

$$\frac{\partial K_{\alpha\beta}}{\partial p_\gamma} = \sum_{e}\frac{\partial K_{e\alpha\beta}}{\partial p_\gamma} \tag{147}$$

$$\frac{\partial K_{e\alpha\beta}}{\partial p_\gamma} = \sum_{k,p,l,q}\frac{\partial K_{ekplq}}{\partial p_\gamma}U_{ekp\alpha}U_{ekp\beta} \tag{148}$$

Local load vector can be calculated using shape functions $N_{ek}(x)$ and load vector t_{ei}

$$Q_{eki} = \int_{\Omega_e} t_{ei}N_{ek}d\Omega \tag{149}$$

Global load vector Q_p can be assembled from the local load vectors Q_{eki}

$$Q_p = \sum_{eki} U_{ekip} Q_{eki} \tag{150}$$

then the sensitivity of the global load vector can be calculated from the sensitivity of the local load vectors

$$\frac{\partial Q_p}{\partial p_l} = \sum_{eki} U_{ekip} \frac{\partial Q_{eki}}{\partial p_l} \tag{151}$$

8.2. Tension-compression problem

The displacement field u in the case of tension-compression problem is described by second order differential equation

$$\frac{d}{dx}\left(EJ\frac{du}{dx}\right) + n = 0 \tag{152}$$

where E is Young modulus, J is a moment of inertia, n is a vector of continuous loads and u is a displacement. After discretization in the case of constant E, A, L we will get the following stiffness matrix

$$K_e = \begin{bmatrix} k_{e11} & k_{e12} \\ k_{e21} & k_{e22} \end{bmatrix} = \begin{bmatrix} \frac{E_e A_e}{L_e} & -\frac{E_e A_e}{L_e} \\ -\frac{E_e A_e}{L_e} & \frac{E_e A_e}{L_e} \end{bmatrix} \tag{153}$$

Sensitivity with respect to the variation of Young modulus

$$\frac{\partial K_e}{\partial E_p} = \begin{bmatrix} \frac{\delta_{ep} A_e}{L_e} & -\frac{\delta_{ep} A_e}{L_e} \\ -\frac{\delta_{ep} A_e}{L_e} & \frac{\delta_{ep} A_e}{L_e} \end{bmatrix} \tag{154}$$

in similar way it is possible to calculate sensitivity with the respect of other parameters. Global stiffness matrix can be calculated in by using the connectivity matrix.
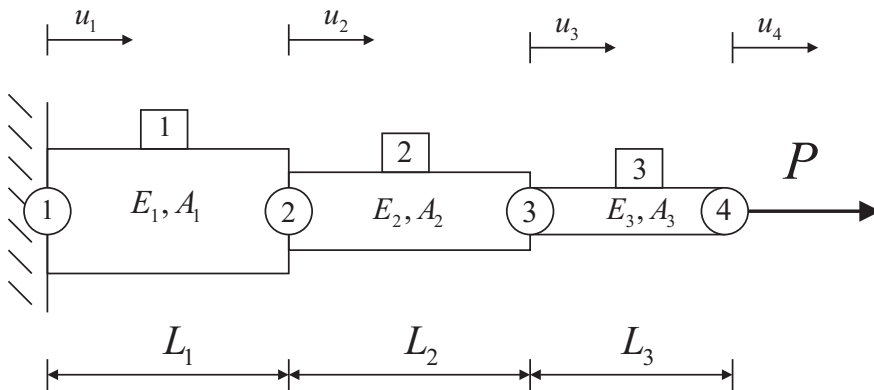

Figure 3. Tension problem

Global stiffness matrix can be calculated in the following way

$$K_1 = \begin{bmatrix} \frac{E_1 A_1}{L_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{155}$$

$$K_2 = \begin{bmatrix} \frac{E_2 A_2}{L_2} & -\frac{E_2 A_2}{L_2} & 0 \\ -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{156}$$

$$K_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{E_3 A_3}{L_3} & -\frac{E_3 A_3}{L_3} \\ 0 & -\frac{E_3 A_3}{L_3} & \frac{E_3 A_3}{L_3} \end{bmatrix} \tag{157}$$

$$K = K_1 + K_2 + K_3 = \begin{bmatrix} \frac{E_1 A_1}{L_1} + \frac{E_2 A_2}{L_2} & -\frac{E_2 A_2}{L_2} & 0 \\ -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} + \frac{E_3 A_3}{L_3} & -\frac{E_3 A_3}{L_3} \\ 0 & -\frac{E_3 A_3}{L_3} & \frac{E_3 A_3}{L_3} \end{bmatrix} \tag{158}$$

Global load vector after applying boundary conditions

$$Q = \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix} \tag{159}$$

Mid point solution is a solution of the following system of equation

$$Ku = Q \tag{160}$$

where

$$u = \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{161}$$

Sensitivity of the displacement u with respect of value of Young modulus E_2 can be calculated from the following system of equation

$$K \frac{\partial u}{\partial E_2} = \frac{\partial Q}{\partial E_2} - \frac{\partial K}{\partial E_2} u \tag{162}$$

where

$$\frac{\partial K}{\partial E_2} = \frac{\partial K_1}{\partial E_2} + \frac{\partial K_2}{\partial E_2} + \frac{\partial K_3}{\partial E_2} = \begin{bmatrix} \frac{A_2}{L_2} & -\frac{A_2}{L_2} & 0 \\ -\frac{A_2}{L_2} & \frac{A_2}{L_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{163}$$

$$\frac{\partial Q}{\partial E_2} = \frac{\partial}{\partial E_2} \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{164}$$

Displacements in the first element

$$u_1(x) = [N_{11}(x), N_{12}(x)] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{165}$$

Sensitivity of the displacements in the first element

$$\frac{\partial u_1(x)}{\partial E_2} = \left[1 - \frac{x}{L_1}, \frac{x}{L_1}\right] \begin{bmatrix} \frac{\partial u_1}{\partial E_2} \\ \frac{\partial u_2}{\partial E_2} \end{bmatrix} \tag{166}$$

Displacements in the second element

$$u_2(x) = [N_{21}(x), N_{22}(x)] \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \tag{167}$$

Sensitivity of the displacements in the first element

$$\frac{\partial u_2(x)}{\partial E_2} = \left[1 - \frac{x}{L_2}, \frac{x}{L_2}\right] \begin{bmatrix} \frac{\partial u_2}{\partial E_2} \\ \frac{\partial u_3}{\partial E_2} \end{bmatrix} \tag{168}$$

Displacements in the third element

$$u_3(x) = [N_{31}(x), N_{32}(x)] \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} \tag{169}$$

Sensitivity of the displacements in the third element

$$\frac{\partial u_3(x)}{\partial E_2} = \left[1 - \frac{x}{L_3}, \frac{x}{L_3}\right] \begin{bmatrix} \frac{\partial u_3}{\partial E_2} \\ \frac{\partial u_4}{\partial E_2} \end{bmatrix} \tag{170}$$

Now it is possible to calculate the strain in all elements

$$\varepsilon_1 = \frac{du}{dx} = \left[-\frac{1}{L_1}, \frac{1}{L_1}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{171}$$

$$\varepsilon_2 = \frac{du}{dx} = \left[-\frac{1}{L_2}, \frac{1}{L_2}\right] \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \tag{172}$$

$$\varepsilon_2 = \frac{du}{dx} = \left[-\frac{1}{L_3}, \frac{1}{L_3}\right] \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} \tag{173}$$

Sensitivity of the strain field

$$\frac{\partial \varepsilon_1}{\partial E_2} = \left[-\frac{1}{L_1}, \frac{1}{L_1}\right] \begin{bmatrix} \frac{\partial u_1}{\partial E_2} \\ \frac{\partial u_2}{\partial E_2} \end{bmatrix} \tag{174}$$

$$\frac{\partial \varepsilon_2}{\partial E_2} = \left[-\frac{1}{L_2}, \frac{1}{L_2}\right] \begin{bmatrix} \frac{\partial u_2}{\partial E_2} \\ \frac{\partial u_3}{\partial E_2} \end{bmatrix} \tag{175}$$

$$\frac{\partial \varepsilon_3}{\partial E_2} = \left[-\frac{1}{L_3}, \frac{1}{L_3} \right] \begin{bmatrix} \frac{\partial u_3}{\partial E_2} \\ \frac{\partial u_4}{\partial E_2} \end{bmatrix} \tag{176}$$

The stress in elements

$$\sigma_1 = E_1 \varepsilon_1 \tag{177}$$

$$\sigma_2 = E_2 \varepsilon_2 \tag{178}$$

$$\sigma_3 = E_3 \varepsilon_3 \tag{179}$$

The sensitivity of the stress filed

$$\frac{\partial \sigma_1}{\partial E_2} = E_1 \frac{\partial \varepsilon_1}{\partial E_2} \tag{180}$$

$$\frac{\partial \sigma_2}{\partial E_2} = 1 \cdot \varepsilon_2 + E_2 \frac{\partial \varepsilon_2}{\partial E_2} \tag{181}$$

$$\frac{\partial \sigma_3}{\partial E_2} = E_3 \frac{\partial \varepsilon_3}{\partial E_2} \tag{182}$$

Above described formulas are true only if the Young modulus $E = E(x)$ and the area of cross-section $A = A(x)$ is constant inside each element. If these functions are not constant then the stiffness matrix have to be calculated by using the integration.

$$K_1 = \begin{bmatrix} \int\limits_0^L E_1(x)A_1(x)\frac{dN_{12}}{dx}\frac{dN_{12}}{dx}dx & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{183}$$

$$K_2 = \begin{bmatrix} \int\limits_0^L E_2(x)A_2(x)\frac{dN_{21}}{dx}\frac{dN_{21}}{dx}dx & \int\limits_0^L E_2(x)A_2(x)\frac{dN_{21}}{dx}\frac{dN_{12}}{dx}dx & 0 \\ \int\limits_0^L E_2(x)A_2(x)\frac{dN_{22}}{dx}\frac{dN_{21}}{dx}dx & \int\limits_0^L E_2(x)A_2(x)\frac{dN_{22}}{dx}\frac{dN_{22}}{dx}dx & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{184}$$

$$K_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \int\limits_0^L E_3(x)A_3(x)\frac{dN_{31}}{dx}\frac{dN_{31}}{dx}dx & \int\limits_0^L E_3(x)A_3(x)\frac{dN_{31}}{dx}\frac{dN_{32}}{dx}dx \\ 0 & \int\limits_0^L E_3(x)A_3(x)\frac{dN_{32}}{dx}\frac{dN_{31}}{dx}dx & \int\limits_0^L E_3(x)A_3(x)\frac{dN_{32}}{dx}\frac{dN_{33}}{dx}dx \end{bmatrix} \tag{185}$$

The functional derivative can be calculated without differentiation.

$$\frac{\delta K_1}{\delta E_2(x)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{186}$$

$$\frac{\delta K_2}{\delta E_2(x)} = \begin{bmatrix} A_2(x)\frac{dN_{21}}{dx}\frac{dN_{21}}{dx} & A_2(x)\frac{dN_{21}}{dx}\frac{dN_{12}}{dx} & 0 \\ A_2(x)\frac{dN_{22}}{dx}\frac{dN_{21}}{dx} & A_2(x)\frac{dN_{22}}{dx}\frac{dN_{22}}{dx} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{187}$$

$$\frac{\delta K_3}{\delta E_2(x)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{188}$$

$$\frac{\delta K}{\delta E_2(x)} = \frac{\delta K_1}{\delta E_2(x)} + \frac{\delta K_2}{\delta E_2(x)} + \frac{\delta K_3}{\delta E_2(x)} \tag{189}$$

$$\frac{\delta Q}{\delta E_2(x)} = \frac{\delta}{\delta E_2(x)} \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{190}$$

Functional derivative of the displacements can be calculated from the following system of equations

$$K \frac{\delta u}{\delta E_2(x)} = \frac{\delta Q}{\delta E_2(x)} - \frac{\delta K}{\delta E_2(x)} u \tag{191}$$

In the same way it is possible to calculate the functional derivative of the displacements, stress and strain fields.

8.3. Truss structures

Using the sensitivity analysis method it is possible to calculate the interval displacements in the truss structures with the interval Young modulus and the area of cross-section (Fig. 4). The struc-
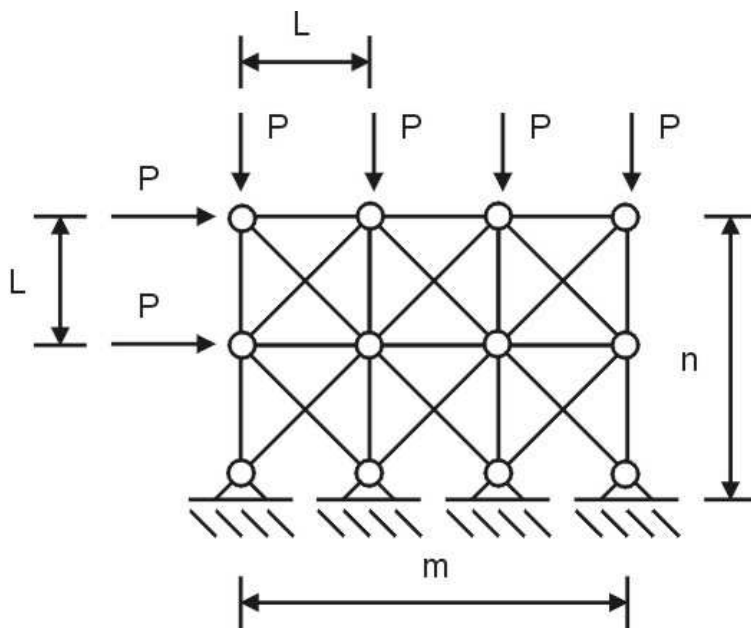
Figure 4. Plain stress-strain problem

ture can be described in two steps. In the first step the truss structure is described by using ANSYS FEM (http://www.ansys.com) program internal scripting language. In this case the uncertainty of

Table I. Interval displacements of truss structure

ID	Interval displacement $[m]$	node	dof
u[0]	[2.54206368e-05, 2.88890319e-05]	5	1
u[1]	[-2.41613231e-06, -5.5936232e-07]	5	2
u[2]	[1.89488493e-05, 2.18240888e-05]	6	1
u[3]	[-1.18336781e-05, -9.68801242e-06]	6	2
u[4]	[1.74375666e-05, 2.00368684e-05]	7	1
u[5]	[-1.53016570e-05, -1.28438219e-05]	7	2
u[6]	[2.23883755e-05, 2.55322229e-05]	8	1
u[7]	[-2.43184098e-05, -2.13175562e-05]	8	2
u[8]	[4.47984203e-05, 5.07482294e-05]	9	1
u[9]	[-1.25873042e-05, -9.13828295e-06]	9	2
u[10]	[3.58319463e-05, 4.09641151e-05]	10	1
u[11]	[-2.03184368e-05, -1.75638790e-05]	10	2
u[12]	[3.30408793e-05, 3.79901925e-05]	11	1
u[13]	[-2.87524495e-05, -2.54594638e-05]	11	2
u[14]	[3.51831538e-05, 4.042328624e-05]	12	1
u[15]	[-4.18322390e-05, -3.7394527e-05]	12	2

the Young modulus is 5% (MP, EX, 1, 5) and the uncertainty of the area of cross-section is also 5% (R, 1, 5). The interval displacements are shown in the Table I.

This example shows that the sensitivity analysis can be use as an extension of existing FEM code.

8.4. PLAIN STRESS

Let us consider a 2D structure which is shown on Fig. 5.

In calculation linear-elastic plain stress-strain mathematical model was used. Young nodulus was uncertain and equal to $E \in [210 \cdot 10^9, 212 \cdot 10^9]\frac{N}{m^2}$, Poisson number $\nu \in [0.2, 0.4]$, thickness $h = 0.1m$, width $L = 1m$, height $h = 1m$ surface load $t_y \in [3, 2]kN$. Numerical results are shown in the Table II. The results are show in the following format

$u[number] = [lower\ bound,\ midpoint\ solution,\ upper\ bound]$.

8.5. INTERVAL STRESS IN 3D ELASTIC BODY

Using described theory it is possible to calculate the interval stress using the 3D brick elements (Fig. 6). Let us consider 6 finite elements with continuous loads $q \in [1, 3]\frac{kN}{m}$, Young modulus $E \in [210, 212]10^9 \frac{N}{m^2}$, Poisson number $\nu \in [0.2, 0.4]$ which are shown in the Fig. 7.

In each element there are 27 Gauss points. Results of calculations are shown in the table below.
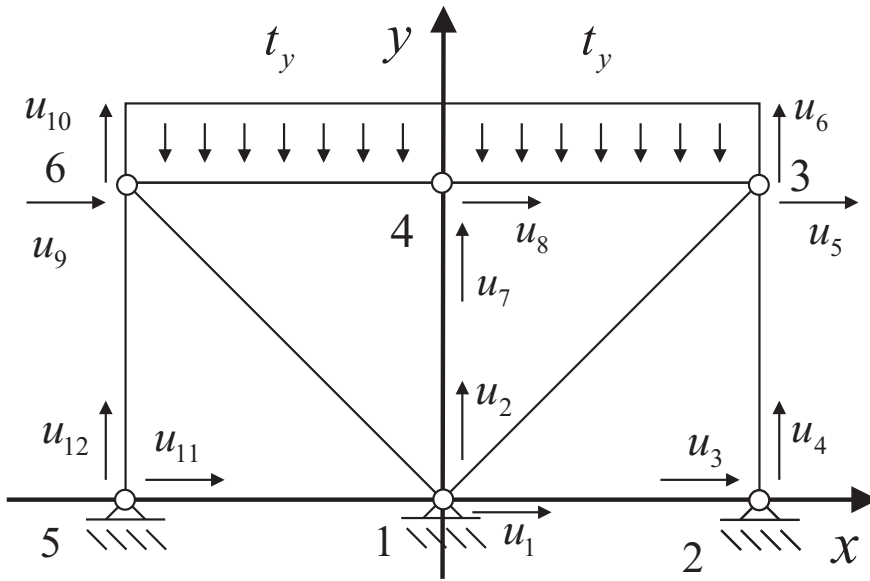
Figure 5. Plain stress-strain problem

Table II. Interval displacements in the truss structure

ID	Interval displacement [m]
u[5]	[7.010160e-08, 9.384325e-08, 1.175510e-07]
u[6]	[-4.461538e-07, -3.479902e-07, -2.587601e-07]
u[7]	[-4.600000e-07, -3.619668e-07, -2.716981e-07]
u[9]	[-1.175510e-07, -9.384325e-08, -7.010160e-08]
u[10]	[-4.461538e-07, -3.479902e-07, -2.587601e-07]

The program can be run from the web page http://andrzej.pownuk.com. The structure is described using some easy to understand scripting language.

9. The computer program

Sensitivity analysis is implemented in object oriented C++ computer program. In the program there is 11 finite elements. The program allow to use the following analysis types

1. Liner static analysis (classical FEM solution)

2. Liner static analysis with interval combinatoric

3. Liner static analysis with sensitivity analysis
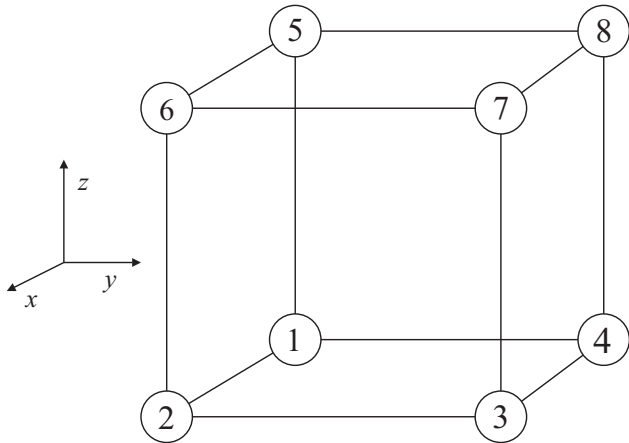
Figure 6. 3D brick element
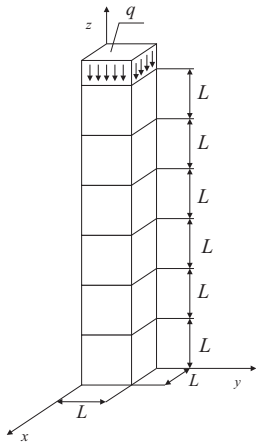
Figure 7. 3D brick elements with surface load

4. Liner static analysis with Taylor expansion method

5. Liner static analysis with functional derivative method

6. Liner static analysis with combination of functional derivative and sensitivity analysis method

The program can be run on-line from the web page http://andrzej.pownuk.com. In the first box there is a description of the problem Fig. 8.
After clicking the button "calculate" the results will appear in the second box.
In order to see all steps of the calculations "debug" command can be apply (e.g. debug_interval_solution).
In order to see the intermediate results commands "print" can be applied (e.g. print_global_stiffness_matrix).

Table III. Interval stress in the element 1

Number of Gauss point	interval stress $\sigma_{zz} \left[\frac{N}{m^2}\right]$
1	[-3.000000e+03, -1.000000e+03]
2	[-3.000000e+03, -1.000000e+03]
3	[-3.000000e+03, -1.000000e+03]
4	[-3.000000e+03, -1.000000e+03]
5	[-3.000000e+03, -1.000000e+03]
6	[-3.000000e+03, -1.000000e+03]
etc.	etc.

Table IV. Interval von Mises stress in the element 1

Number of Gauss point	interval von Mises stress $\sigma_M \left[\frac{N}{m^2}\right]$
1	[1.000000e+03, 3.000000e+03]
2	[1.000000e+03, 3.000000e+03]
3	[1.000000e+03, 3.000000e+03]
4	[1.000000e+03, 3.000000e+03]
5	[1.000000e+03, 3.000000e+03]
6	[1.000000e+03, 3.000000e+03]
etc.	etc.
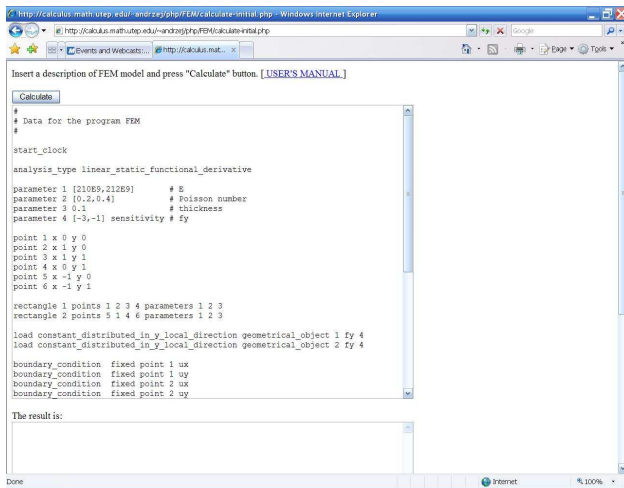
Figure 8. Web application

Figure 9. Results of the calculations

10. Interval eigenvalue

10.1. SENSITIVITY OF THE EIGENVALUES

In the dynamical problems of structural mechanics the finite element method lead to the following system of differential equations

$$M\ddot{u} + Ku = 0 \tag{192}$$

If we assume that the solution is in the following form

$$u = u_0 \sin(\omega t + \phi) \tag{193}$$

then

$$\dot{u} = \omega u_0 \cos(\omega t + \phi), \quad \ddot{u} = -\omega^2 u_0 \sin(\omega t + \phi) \tag{194}$$

and from the equation (192) we have

$$-M\omega^2 u_0 \sin(\omega t + \phi) + Ku_0 \sin(\omega t + \phi) = 0 \tag{195}$$

$$(K - \omega_j^2 M)u_j = 0 \tag{196}$$

Eigenvectors $u_1, ..., u_n$ are M-orthogonal

$$u_i^T M u_j = \delta_{ij} \tag{197}$$

then from the equation (196)

$$u_i^T K u_j = \omega_j^2 \delta_{ij} \tag{198}$$

Sensitivity with the respect to the parameter p

$$\left(\frac{\partial K}{\partial p} - \frac{\partial \omega_j^2}{\partial p} M - \omega_j^2 \frac{\partial M}{\partial p} \right) u_j + (K - \omega_j^2 M) \frac{\partial u_j}{\partial p} = 0 \tag{199}$$

Lets multiply above equation by u_i^T

$$u_i^T \left(\frac{\partial K}{\partial p} - \frac{\partial \omega_j^2}{\partial p} M - \omega_j^2 \frac{\partial M}{\partial p} \right) u_j = 0 \tag{200}$$

$$u_i^T \left(\frac{\partial K}{\partial p} - \omega_j^2 \frac{\partial M}{\partial p} \right) u_j = \frac{\partial \omega_j^2}{\partial p} u_i^T M u_j \tag{201}$$

$$u_i^T \left(\frac{\partial K}{\partial p} - \omega_j^2 \frac{\partial M}{\partial p} \right) u_j = \frac{\partial \omega_j^2}{\partial p} \delta_{ij} \tag{202}$$

Then sensitivity of the frequency of vibration ω_j^2 can be calculated from the following formula (Lund, 1994; Hilbert and Courant, 1953)

$$\frac{\partial \omega_j^2}{\partial p} = u_j^T \left(\frac{\partial K}{\partial p} - \omega_j^2 \frac{\partial M}{\partial p} \right) u_j \tag{203}$$

The interval frequency of vibration can be calculated using sensitivity analysis and derivative $\frac{\partial \omega_j}{\partial p}$.

$$\frac{\partial \omega_j^2}{\partial p} = 2\omega_j \frac{\partial \omega_j}{\partial p} \tag{204}$$

$$\frac{\partial \omega_j}{\partial p} = \frac{1}{2\omega_j} \frac{\partial \omega_j^2}{\partial p} = \frac{1}{2\omega_j} u_j^T \left(\frac{\partial K}{\partial p} - \omega_j^2 \frac{\partial M}{\partial p} \right) u_j \tag{205}$$

If the derivative of stiffness matrix $\frac{\partial K}{\partial p}$ and the mass matrix $\frac{\partial M}{\partial p}$ are constant then the sign of the derivative $\frac{\partial \omega_j^2}{\partial p}$ is constant and extreme values of ω^2 can be calculated by using sensitivity analysis. Let us consider the system of first order differential equation in the matrix form

$$\dot{x} = Ax \tag{206}$$

If we assume that the solution has the following form $x = x_0 e^{\lambda t}$, $x = \lambda x_0 e^{\lambda t}$ then

$$\lambda x_0 e^{\lambda t} = A x_0 e^{\lambda t}, \quad \Rightarrow \quad (A - \lambda I)x_0 = 0 \tag{207}$$

Then we have the standard eigenvalue problem. Derivative with respect of parameter p is equal to the following

$$\left(\frac{\partial A}{\partial p} - \frac{\partial \lambda_j}{\partial p} I \right) x_j + (A - \lambda_j I) \frac{\partial x_j}{\partial p} = 0 \tag{208}$$

$$x_i^T \left(\frac{\partial A}{\partial p} - \frac{\partial \lambda_j}{\partial p} I \right) x_j = 0, \quad \Rightarrow \quad \frac{\partial \lambda_j}{\partial p} x_i^T x_j = x_i^T \frac{\partial A}{\partial p} x_j \tag{209}$$

Finally derivative of the eigenvalue can be calculated from the following formula

$$\frac{\partial \lambda_j}{\partial p} = x_j^T \frac{\partial A}{\partial p} x_j \tag{210}$$

Now it is possible to apply sensitivity analysis method in order to calculate upper and lower bound of the eigenvalue λ_j.

If the derivative of the matrix A i.e. $\frac{\partial A}{\partial p}$ is constant then the sign of the derivative $\frac{\partial \lambda}{\partial p}$ is constant and extreme values of λ can be calculated by using sensitivity analysis.

Different method which is based on perturbation of positive definite matrices is described in the paper (Modares, Mullen and Muhanna, 2006).

10.2. VIBRATION OF MULTIBODY SYSTEM

Dynamics of the mechanical system, which is shown in the Fig. 10 is described by the following system of differential equation
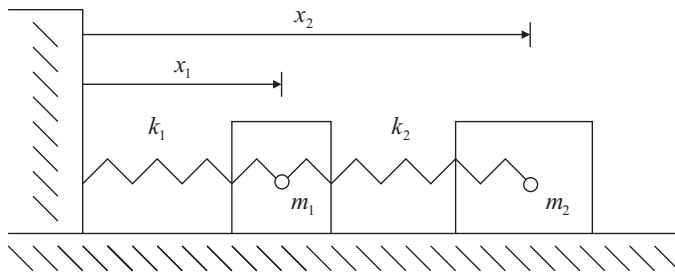

Figure 10. Multibody system

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{211}$$

or shortly

$$M\ddot{x} + Kx = 0 \tag{212}$$

where $k = k_1 = k2$ and $m = m_1 = m_2$. The eigenvalue problem

$$det\left(K - \omega^2 M\right) = 0 \tag{213}$$

has the following solution

$$\omega_1 = \sqrt{(3 - \sqrt{5})\frac{k}{2m}}, \quad \omega_2 = \sqrt{(3 + \sqrt{5})\frac{k}{2m}} \tag{214}$$

The eigenvectors x_1, x_2 satisfy the following system of linear equations

$$(K - \omega_1^2 M)x_1 = 0 \tag{215}$$

$$x_1 = \begin{bmatrix} \frac{\sqrt{5}-1}{\sqrt{2(5-\sqrt{5})m}} \\ \frac{\sqrt{2}}{\sqrt{(5-\sqrt{5})m}} \end{bmatrix} \tag{216}$$

$$(K - \omega_2^2 M)x_2 = 0 \tag{217}$$

$$x_2 = \begin{bmatrix} -\dfrac{\sqrt{5}+1}{\sqrt{2(5+\sqrt{5})m}} \\[3mm] \dfrac{\sqrt{2}}{\sqrt{(5+\sqrt{5})m}} \end{bmatrix} \tag{218}$$

$$\frac{\partial \omega_1^2}{\partial m} = x_1^T \left(\frac{\partial K}{\partial m} - \omega_1^2 \frac{\partial M}{\partial m} \right) x_1 = -(3 - \sqrt{5})\frac{k}{2m^2} < 0 \tag{219}$$

$$\frac{\partial \omega_2^2}{\partial m} = x_2^T \left(\frac{\partial K}{\partial m} - \omega_2^2 \frac{\partial M}{\partial m} \right) x_2 = -(3 + \sqrt{5})\frac{k}{2m^2} < 0 \tag{220}$$

$$\frac{\partial \omega_1^2}{\partial k} = x_1^T \left(\frac{\partial K}{\partial k} - \omega_1^2 \frac{\partial M}{\partial k} \right) x_1 = (3 - \sqrt{5})\frac{1}{2m} > 0 \tag{221}$$

$$\frac{\partial \omega_2^2}{\partial k} = x_2^T \left(\frac{\partial K}{\partial k} - \omega_2^2 \frac{\partial M}{\partial k} \right) x_2 = (3 + \sqrt{5})\frac{1}{2m} > 0 \tag{222}$$

If we assume that the sign of the eigenvalue is constant, then extreme values of the eigenvalues can be calculated in the following way

$$\underline{\omega}_1 = \omega_1(\overline{m}, \underline{k}), \quad \overline{\omega}_1 = \omega_1(\underline{m}, \overline{k}) \tag{223}$$

$$\underline{\omega}_2 = \omega_2(\overline{m}, \underline{k}), \quad \overline{\omega}_2 = \omega_2(\underline{m}, \overline{k}) \tag{224}$$

where $m \in [\underline{m}, \overline{m}]$, $k \in [\underline{k}, \overline{k}]$.

11. Conclusions

Using functional derivative it is possible to check monotonicity of the function with uncertain functional parameters. If the function $u = u(p)$ is monotone then extreme values of the results can be calculated by using upper and lower bound of the functional intervals and sensitivity analysis (Neumaier and Pownuk, 2004; Pownuk, 2004). Sensitivity can be use as an extension of the existing FEM programs. Using quasi analytical method it is possible to avoid approximation errors. Functional derivative can be sometimes calculated without integration. This property may increase accuracy of the solution. Using the sensitivity analysis method it is possible to calculate the interval eigenvalues. Interval eigenvalues can be calculated also in the case of structures with uncertain shape and uncertain functional parameters.

Presented sensitivity analysis method can be applied to the solution of any problem with functional parameters in which it is possible to calculate the functional derivative and verify monotonicity. For non-monotone problems it is possible to apply an extension of the algorithm, which gives only inner bounds.

The approach presented can be applied together with any numerical method for the solution of the underlying problem, including techniques for partial differential equations e.g. FEM, FDM, BEM, FVM etc. Extended version of this paper was published as a research report at the web page of the University of Texas at El Paso (Pownuk, 2007).

References

M.W. Hirsch and H. Smith. Monotone maps: a review. *Journal of Difference Equations and Applications*, 11(4-5):379–398, 2005.

E. Lund. *Finite Element Based Design Sensitivity Analysis and Optimization*. Ph.D. Dissertation, Aalborg University, Aalborg, 1994.

M. Modares, R. L. Mullen, and R.L. Muhanna. Natural frequencies of a structure with bounded uncertainty. *Journal of Engineering Mechanics, ASCE*, 132, 2006.

D. Moens and D. Vandepitte. A survey of non-probabilistic uncertainty treatment in finite element analysis. *Computer Methods in Applied Mechanics and Engineering*, 194(14-16), 2005.

R.E. Moore. *Interval Analysis*. Prentice-Hall, Englewood Cliffs, 1966.

A. Neumaier. *Interval Methods for Systems of Equations*. Cambridge Univ. Press, Cambridge, 1990.

A. Neumaier and A. Pownuk. Linear systems with large uncertainties with applications to truss structures. *Reliable Computing*, 13(2):149–172, 2007.

A. Pownuk. Numerical solutions of fuzzy partial differential equation and its application in computational mechanics. *Fuzzy Partial Differential Equations and Relational Equations: Reservoir Characterization and Modeling, (M. Nikravesh, L. Zadeh and V. Korotkikh, eds.), Studies in Fuzziness and Soft Computing, Physica-Verlag*, pages 308–347, 2004.

A. Pownuk. Numerical solution of fem equations with uncertain functional parameters. *FEMTEC 2006, The University of Texas at El Paso, December 11 - 15, El Paso, USA*, 2006.

A. Pownuk, General Interval FEM Program Based on Sensitivity Analysis *The University of Texas at El Paso*, Department of Mathematical Sciences Research Reports Series Texas Research Report No. 2007-06, El Paso, Texas, USA, http://www.math.utep.edu/preprints/2007/2007-06.pdf

D. Hilbert R. Courant. *Methods of Mathematical Physics, Vol. 1.* Interscience Publishers, New York, 1953.

O.C. Zienkiewicz and R.L. Taylor. *The Finite Element Method. Fifth edition. Volume 1: Basis*, volume 1. Butterworth Heinemann, London, 5 edition, 2000.